Summary:
- We give an additional formula for the coefficients in a Laurent series expansion, and discuss how to determine the region of convergence of the series.
- We then discuss poles, essential singularities, and zeros, based on Taylor and Laurent series expansions.
- We define residues, and state the residue theorem.

(Goursat, §§40 – 43.)

24. Laurent series, revisited. Recall from before the term test that if we have a function f, analytic between two circles both centred at a, say C and C', with C' contained inside C, then if we define

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz' \quad (n \geq 0), \qquad b_n = \frac{1}{2\pi i}\int_{C'} (z'-a)^{n-1} f(z')\,dz', \quad (n \geq 1) \tag{1}$$

we have the following series expansion for f:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-a)^n}, \tag{2}$$

and the series on the right-hand side converge on the annular region between C and C'. Now since f is analytic between C and C', as are the quantities $(z'-a)^{-(n+1)}$ and $(z'-a)^{n-1}$ in the definitions of a_n and b_n, the Cauchy integral theorem allows us to evaluate a_n and b_n over *any* simple closed curve γ which lies entirely in the annular region between C and C'. Suppose that we therefore replace C and C' in (1) by γ. Then, noting that we may write

$$b_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{(z'-a)^{-n+1}}\,dz',$$

we see that if we define

$$J_n = \begin{cases} a_n, & n \geq 0, \\ b_{-n}, & n < 0, \end{cases}$$

we see that we have for all $n \in \mathbf{Z}$, positive and negative,

$$J_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{(z'-a)^{n+1}}\,dz', \tag{3}$$

and moreover that we may write the series expansion for f given above as

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} J_n (z-a)^n + \sum_{n=1}^{\infty} J_{-n}(z-a)^{-n} \\ &= \sum_{n=0}^{\infty} J_n (z-a)^n + \sum_{n=-1}^{-\infty} J_n (z-a)^n = \sum_{n=-\infty}^{\infty} J_n (z-a)^n, \end{aligned} \tag{4}$$

where this last sum is essentially defined by the expression on the right-hand side of the first line.[1]

There is no essential difference between the expansions (2) and (4); they are just different ways of expressing the same information. On the other hand, the expression (3) is certainly more symmetric and probably easier to remember than the expression (1). We pay for this in some sense, though, by the fact that the expansion (4) in some sense hides the singular terms, and we must keep in mind that if n is negative the terms $(z-a)^n$ are singular at $z = a$.

25. Region of convergence. There are various ways of determining the region of convergence of a power or Laurent series. Perhaps the most elementary way is to apply the root test we learned in elementary

[1] Note that this is how we define improper integrals of the form $\int_{-\infty}^{+\infty} f(x)\,dx$ in elementary calculus: we pick some point $a \in \mathbf{R}$ and define the integral to be the sum $\int_{-\infty}^{a} f(x)\,dx + \int_a^{+\infty} f(x)\,dx$, much as we may define this last series as $\sum_{n=-\infty}^{-1} J_n(z-a)^n + \sum_{n=0}^{\infty} J_n(z-a)^n$.

calculus. Let us first recall the root test for a series of positive real numbers: if $c_n \geq 0$ for all n, then the series

$$\sum_{n=0}^{\infty} c_n$$

will converge if the quantity $\lim_{n\to\infty} c_n^{1/n} < 1$ and diverge if $\lim_{n\to\infty} c_n^{1/n} > 1$; if $\lim_{n\to\infty} c_n^{1/n} = 1$ then the test is indeterminate and we must do something else. Now suppose that we have a Taylor series

$$\sum_{n=0}^{\infty} a_n(z-a)^n.$$

This series will converge if it is *absolutely convergent*, i.e., if the series of absolute values

$$\sum_{n=0}^{\infty} |a_n(z-a)^n|$$

converges. If we apply the root test to this series, we see that the series is convergent if

$$\lim_{n\to\infty} |a_n(z-a)^n|^{1/n} = |z-a| \lim_{n\to\infty} |a_n|^{1/n} < 1$$

and divergent if

$$\lim_{n\to\infty} |a_n(z-a)^n|^{1/n} = |z-a| \lim_{n\to\infty} |a_n|^{1/n} > 1.$$

Let us now define the quantity R by

$$R^{-1} = \lim_{n\to\infty} |a_n|^{1/n};$$

if this limit is zero we set $R = \infty$, while if the limit is infinite we set $R = 0$. Then from the foregoing we see that the series will converge if

$$|z-a| < R$$

and diverge if

$$|z-a| > R.$$

(If $|z-a| = R$, the root test fails and the series may either converge or diverge.)

Now suppose that we consider instead the Laurent series

$$\sum_{n=1}^{\infty} b_n(z-a)^{-n};$$

by exactly the same logic, if in this case we define R' by

$$R' = \lim_{n\to\infty} |b_n|^{1/n},$$

then we see that the series will converge if

$$|z-a|^{-1} < \frac{1}{R'}$$

and diverge if

$$|z-a|^{-1} > \frac{1}{R'};$$

in other words, it will converge if

$$|z-a| > R'$$

and diverge if

$$|z-a| < R'.$$

(As usual, the test fails when $|z - a| = R'$, meaning that it tells us nothing about the convergence or divergence of the series.) This means that, while Taylor series converge on disks, the singular part of a Laurent series converges instead on the *exterior* of a disk. The full Laurent series, being the sum of a Taylor series and a singular part, will converge on the intersection of one disk with the exterior of another disk, i.e., on an annulus (exactly as we might expect!).[2]

The preceding method will allow us to find the region where any given Taylor or Laurent series converges, assuming that we can calculate the two limits involved. Thus it is useful when the only thing we know is the series itself. On the other hand, if we know the function f to which the series converges, then there is a much simpler method, as follows. Let us consider Laurent series; the same logic applies to Taylor series (which are, after all, just Laurent series without no singular part, i.e., with singular part equal to 0). Suppose that f is a function which is analytic between the circles C' and C, centred at a (with C' inside of C as usual). Then our results with Laurent series show that the Laurent series of f converges to f everywhere on this annulus. Now if f were analytic on a larger annulus, say between circles C_1' and C_1, also centred at a, then its Laurent series on the annulus between C_1' and C_1 would converge between C_1' and C_1; but by the Cauchy integral theorem (for example, recall that we can define the coefficients using any curve γ lying in the annular region) the coefficients of this Laurent series would be the same as those of the original Laurent series. Thus the original Laurent series must also converge on this larger annulus.

If we take this reasoning to its logical conclusion, we see that the Laurent series will converge on the largest annular region on which f is analytic. To be a bit more careful, recall that we need f to be continuous on the boundary curves C and C'; thus the Laurent series will converge on the largest annular region such that f is analytic in the interior and continuous on the boundary. This means that, if we are attempting to find where the Laurent series for a *given* function f is convergent, we need only determine the points where f is not analytic; the Laurent series will then converge on the largest annular region, centred at a, which does not contain any of these points.

(It is worth pointing out that this method does *not* apply to the case where we are simply given a Laurent series and do not know the function to which it converges. This is because it is in general not possible to determine the points where a function is not analytic simply by examining its Laurent series.)

Let us give an example.

EXAMPLE. Let us define a function f by

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} z^{-n}.$$

We will find where these series converge, and then discuss how to use the formulas above to determine its Laurent series. (In a certain sense this second part is quite pointless since a Laurent series is unique, so the series expansion on the right-hand side *is* the Laurent expansion of f around 0. But on the other hand it will give us some practice in the use of the general formulas.) First of all, it can be shown that

$$\lim_{n\to\infty} (n!)^{1/n} = \infty;$$

thus also

$$\lim_{n\to\infty} [(n!)^2]^{1/n} = \infty,$$

and so the first series must converge on the entire complex plane.

Now it can be shown, using L'Hôpital's rule, that

$$\lim_{n\to\infty} n^{1/n} = 1,$$

[2] Note that it is, in principle, quite possible that this intersection might be empty, i.e., that the disk *outside* of which the singular part converges is larger than the disk *inside* of which the analytic part converges. In this case, the series simply does not converge at any point in the complex plane. If we obtained this result by starting from some specific function, it would indicate that the function was not analytic on any annular region centred at a. (This does not, incidentally, mean that the function is never analytic, just that the region on which it is analytic does not contain any annulus around a.)

whence

$$\lim_{n\to\infty}[n^2]^{1/n} = 1,$$

and the quantity R' above will be 1, meaning that the second series converges when

$$|z| > 1.$$

We pause here for a moment to discuss the relation of this last result to our second method for determining the region of convergence of a Laurent series. Let us define

$$g(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} z^{-n}$$

wherever this series converges. Since the series diverges for $|z| < 1$, the function g must have some singularity on the unit circle. Now the series is clearly absolutely convergent on the unit circle itself (since if $|z| = 1$ the absolute value of the terms of the series is simply $1/n^2$, giving a convergent series); thus g does not diverge at any point on the unit circle. Let us look at its derivative. Differentiating term-by-term, we obtain

$$g'(z) = \sum_{n=1}^{\infty} -\frac{z^{-n-1}}{n};$$

thus

$$zg'(z) = \sum_{n=1}^{\infty} -\frac{z^{-n}}{n},$$

so

$$\frac{d}{dz}[zg'(z)] = \sum_{n=1}^{\infty} z^{-n-1} = z^{-2}\sum_{n=0}^{\infty} z^{-n} = z^{-2}\frac{1}{1-z^{-1}} = \frac{1}{z^2 - z} = \frac{1}{z(z-1)}.$$

Now this function clearly has a singularity at $z = 1$, which is on the unit circle. While it is not entirely clear how to determine the full function g given this rather peculiar differential operator on g, we can say that if g were analytic at $z = 1$, then so would be $d/dz[zg'(z)]$; since this latter quantity, as just noted, is *not* analytic at $z = 1$, g cannot be analytic there either. (Some further study suggests that g in fact has a branch point at $z = 1$, but we shall not show that here.) This explains why the series cannot converge on a larger annulus.

Proceeding, let us see how to calculate the coefficients in the Laurent series expansion for f using the formula above. (This is very similar to, though more complicated than, the example we did right before the break.) We shall use the formula (3) with $a = 0$, and γ any curve contained in the region $|z| > 1$:

$$J_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{z'^{\,n+1}}\,dz'.$$

Substituting in the series definition of f, we have

$$J_n = \frac{1}{2\pi i}\int_\gamma \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\frac{z'^{\,k}}{z'^{\,n+1}} + \sum_{k=1}^{\infty}\frac{1}{k^2}\frac{1}{z'^{,k+n+1}}\,dz',$$

so that if we assume we can interchange sum and integral, then we only need to evaluate the integrals

$$\int_\gamma z'^{\,k-n-1}\,dz', \qquad \int_\gamma z'^{-k-n-1}\,dz'.$$

It is sufficient to determine

$$\int_\gamma z'^{\,m}\,dz'$$

where m is any integer (positive or negative). Now if $m \geq 0$, then the integrand will be analytic everywhere on the plane (as usual, we set $z^0 = 1$ for all z by convention). Now suppose that $m \leq -1$, and write $n = -m - 1 \geq 0$; then by the general Cauchy integral formula

$$\int_\gamma z'^m \, dz' = \int_\gamma \frac{1}{z'^{-m}} \, dz' = \int_\gamma \frac{1}{z'^{n+1}} \, dz'$$
$$= 2\pi i \frac{d^n}{dz^n}[1]|_{z=0},$$

which will be $2\pi i$ if $n = 0$ and 0 if $n > 0$, since the derivative of a constant is 0. Pulling all of this together, then, we see that

$$\int_\gamma z'^m \, dz' = \begin{cases} 0, & m \neq -1 \\ 2\pi i, & m = -1. \end{cases}$$

(This is a very useful formula to keep in mind, in general.) Applying this to the above formula for J_n, we see that for $n \geq 0$

$$J_n = \frac{1}{2\pi i} \sum_{k=0}^\infty \frac{1}{(k!)^2} \int_\gamma z'^{k-n-1} \, dz' + \frac{1}{2\pi i} \sum_{k=1}^\infty \frac{1}{k^2} \int_\gamma z'^{-k-n-1} \, dz'$$
$$= \frac{1}{(n!)^2},$$

since the first integral will be nonzero only when $k - n - 1 = -1$, i.e., $k = n$, in the which case it equals $2\pi i$, while the second will be nonzero only when $-k - n - 1 = -1$, i.e., when $k = -n$; since in the second series $k \geq 1$, while $n \geq 0$, this is impossible. Similarly, when $n \leq -1$

$$J_n = \frac{1}{2\pi i} \sum_{k=0}^\infty \frac{1}{(k!)^2} \int_\gamma z'^{k-n-1} \, dz' + \frac{1}{2\pi i} \sum_{k=1}^\infty \frac{1}{k^2} \int_\gamma z'^{-k-n-1} \, dz'$$
$$= \frac{1}{(-n)^2},$$

by the foregoing. Thus we may write

$$J_n = \begin{cases} \frac{1}{(n!)^2}, & n \geq 0, \\ \frac{1}{n^2}, & n \leq -1, \end{cases}$$

which is exactly the coefficients in the original series (as advertised).

26. Isolated singularities. We will now apply Laurent series to study the ways in which functions can fail to be analytic at a single point. More precisely, suppose that $a \in \mathbf{C}$, and let f be a function whose domain contains a. If f is analytic at a, then a is called a *regular point* of f; if f is not analytic at a, then a is called a *singular point* of f. Now suppose that a is a singular point, but that f is analytic everywhere else near a; i.e., that there is some $r > 0$ such that f is analytic on the punctured disk

$$\{z \,|\, 0 < |z - a| < r\},$$

i.e., f is analytic everywhere on the disk of radius r centred at a, except of course at a itself. In this case a is called an *isolated singular point* of f. (Note that branch points are *not* isolated singular points.)

Let us see what we can learn about isolated singular points from Laurent series. Suppose that a is an isolated singular point of a function f, and that f is analytic on the punctured disk

$$\{z \,|\, 0 < |z - a| < r\}.$$

Then on any annular region contained in this punctured disk we will have the Laurent series expansion

$$f = \sum_{n=0}^\infty a_n (z - a)^n + \sum_{n=1}^\infty b_n (z - a)^{-n},$$

where a_n and b_n are given by the formulas (1) above, and moreover do not depend on the particular choice of annular region; thus we may consider this to be an expansion for f on the entire punctured disk (which is, strictly speaking, *not* an annular region in the sense in which we have been using that term). Let us consider the singular part of this expansion, namely

$$\sum_{n=1}^{\infty} b_n (z-a)^{-n}.$$

There are three possibilities: (i) $b_n = 0$ for all n; (ii) $b_n \neq 0$ for only finitely many n; (iii) $b_n \neq 0$ for infinitely many n. In the first case, it can be shown that by defining $f(a) = a_0$, the function f can be made analytic on the whole disk $\{z \,|\, |z-a| < r\}$, so that the 'singularity' is not really a singularity at all; this is called a 'removable singularity'. In the second case we say that the function has a *pole* at a, while in the third case we say that it has an *essential singularity* at a.

Let us give a couple examples.

EXAMPLES. 1. Let $f = 1/\sin z$; then clearly f is analytic everywhere except where $\sin z = 0$, i.e., everywhere except for $z = n\pi$, $n \in \mathbf{Z}$. Each of these points must therefore be an isolated singularity of f. Let us consider the isolated singularity at $z = 0$, and see if we can determine the Laurent series for f there. Now we have

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = z \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}.$$

Now the final series above must converge on the entire complex plane; essentially, this is because it converges at $z = 0$, and other than at that point it is equal to a function analytic on the punctured plane $\{z \,|\, z \neq 0\}$. (If you want to use this fact in your homework solutions, you need to write in a bit more detail!) Let us denote this function by $\phi(z)$, i.e.,

$$\phi(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}.$$

Then since $\phi(0) = 1 \neq 0$, there must be some disk D around 0 on which $\phi \neq 0$. On this disk, then, $1/\phi$ must be analytic, and hence can be expanded as

$$\frac{1}{\phi(z)} = \sum_{k=0}^{\infty} a_k z^k$$

for some set of coefficients a_k, which we could determine by division of series but won't. Thus f can be written as

$$f(z) = \frac{1}{z\phi(z)} = \frac{1}{z} \sum_{k=0}^{\infty} a_k z^k = \frac{a_0}{z} + \sum_{k=0}^{\infty} a_{k+1} z^k.$$

Thus $z = 0$ is a pole for f. We say that it is a *pole of order 1*; we shall define this in general momentarily.

2. Now let us consider the function $f(z) = e^{1/z}$. Clearly, f is analytic everywhere except at $z = 0$, so that 0 is an isolated singular point of f. Now for any complex number z we have

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k;$$

thus for any nonzero complex number z we must have

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k},$$

and this must therefore be the Laurent expansion for f around 0. Since f has infinitely many nonzero coefficients in its singular part, 0 must be an essential singularity for f.

To return to our general treatment, suppose that a is a pole of a function f; this means that a is an isolated singularity of f, and that the singular part of the Laurent expansion of f around a has finitely many nonzero coefficients. Suppose that m is the largest integer for which $b_m \neq 0$, i.e., that $b_m \neq 0$ while $b_k = 0$ for all $k > m$; then we say that a is a pole *of order* m of f. This explains our terminology in the first example above.

In other words, a function f has a pole of order m at a if near a

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \cdots + \frac{b_m}{(z-a)^m},$$

and moreover $b_m \neq 0$.

There is a nice relationship between poles and zeros. Recall that a polynomial $p = a_0 + \cdots + a_n z^n$ is said to have a *zero of order* m at a point a if it is divisible by $(z-a)^m$, i.e., if there is another polynomial q such that $q(a) \neq 0$ and

$$p(z) = (z-a)^m q(z).$$

Now evidently this same definition can be applied to Taylor series. Specifically, suppose that a function f is analytic near a point a; then we may expand it in its Taylor series about a as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

Now f is said to have a *zero of order* m at a if there is an analytic function ϕ near a such that $\phi(a) \neq 0$ and

$$f(z) = (z-a)^m \phi(z).$$

In this case, it is easy to see that the first m terms of the Taylor series for f must all vanish, i.e., that we must have

$$f(z) = a_m (z-a)^m + a_{m+1} (z-a)^{m+1} + \cdots .$$

Now consider $1/f(z)$ near $z = a$; by the foregoing, we have

$$\frac{1}{f(z)} = \frac{1}{(z-a)^m \phi(z)} = \frac{1}{\phi(z)} (z-a)^{-m};$$

now since $\phi(a) \neq 0$, as we saw in Example 1 above, $\phi(z)$ must be nonzero on some disk centred at a; thus $1/\phi(z)$ must be analytic on this disk, say with power series

$$\frac{1}{\phi(z)} = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

and moreover $c_0 = \frac{1}{\phi(a)} \neq 0$. Thus

$$\begin{aligned} \frac{1}{f(z)} &= \frac{1}{(z-a)^m} \sum_{n=0}^{\infty} c_n (z-a)^n \\ &= \sum_{n=0}^{\infty} c_{n+m} (z-a)^n + \frac{c_{m-1}}{z-a} + \cdots + \frac{c_m}{(z-a)^m}, \end{aligned}$$

which shows that $1/f$ must have a pole of order m at $z = a$. This logic works to show the reverse implication also, namely that if a function g has a pole of order m at $z = a$, then $1/g$ must have a zero of order m at $z = a$. Thus, in some sense, poles and zeroes are complementary to each other.

27. Residues. Suppose that a is an isolated singular point of the function f, and let γ be a simple closed curve in the punctured disk about a on which f is analytic, and which moreover encloses the point a. Then we may write the Laurent series for f about a,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}.$$

Thus, assuming that we can interchange sum and integral, we have

$$\int_\gamma f(z')\,dz' = \sum_{n=0}^{\infty} a_n \int_\gamma (z'-a)^n\,dz' + \sum_{n=1}^{\infty} b_n \int_\gamma (z'-a)^{-n}\,dz';$$

by the same logic we used in the examples in the previous section, all of these integrals will vanish except for the one with the power $(z'-a)^{-1}$, and that one will give $2\pi i$. Thus we have

$$\int_\gamma f(z')\,dz' = 2\pi i b_1.$$

The quantity b_1, which is the coefficient of the $(z'-a)^{-1}$ term in the Laurent expansion of f about $z = a$, is called the *residue* of f at the point a.

At present, this logic might seem slightly circular, since the equation above is actually equivalent to the definition of b_1 given above. If we had no other way to find Laurent series than through the definitions of a_n and b_n in terms of integrals, then this would indeed be circular. However, as our examples above have hopefully suggested, there are ways of computing Laurent series that do not involve calculating integrals at all. This means that there are other methods of computing residues. These methods allow us to apply the formula above in meaningful ways.

We may generalise this result as follows to obtain an even more useful one. Suppose that we begin with a curve γ, and that f is analytic everywhere inside γ except at certain isolated singular points $z_1, \cdots, z_n$. Let β_j denote the residue of f around z_j, for $j = 1, \cdots, n$. Then it follows from the generalised Cauchy theorem that we may write

$$\int_\gamma f(z')\,dz' = 2\pi i \sum_{j=1}^{n} \beta_j.$$

This result, known as the *residue theorem*, is extremely useful in the applications we shall make of contour integrals to evaluating definite integrals on the real line.

Summary:
- We give examples of applications of the residue theorem to the evaluation of definite integrals on the real axis.
- We then give theorems for showing that integrals over semicircles go to zero, and provide additional methods for computing residues.

(Goursat, §§44 – 46.)

28. Evaluation of definite integrals. Recall the residue theorem from last time: if f is a function which is continuous on a simple closed curve γ, and analytic inside γ except potentially at a finite number of isolated singularities $z_1, \cdots, z_n$, at which it has residues $\beta_1, \cdots, \beta_n$, respectively. Then we have

$$\int_\gamma f(z')\,dz' = 2\pi i \sum_{j=1}^{n} \beta_j.$$

Let us see by way of an example – which could also be done by elementary methods – how this result can be applied to evaluate definite integrals.

EXAMPLE. Consider the integral

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx.$$

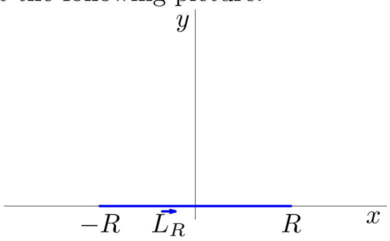
Since this integral converges, it is equal to the limit

$$\lim_{R\to+\infty} \int_{-R}^{R} \frac{1}{1+x^2}\,dx;$$

since $\arctan x$ is an antiderivative of $1/(1+x^2)$, by the fundamental theorem of calculus this integral is equal to

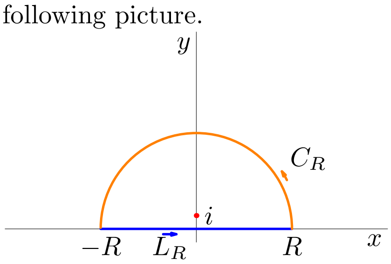
$$\lim_{R\to+\infty} (\arctan R - \arctan(-R)) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Now suppose that we consider $\int_{-R}^{R} \frac{1}{1+x^2}\,dx$ as a contour integral in the complex plane, with the contour taken along the real axis; then we get the following picture.



As it stands this does not seem to have gotten us anywhere. Note though that the integrand here is analytic on the entire plane except for (simple) poles at $\pm i$. Thus if it were possible to somehow *augment* the contour L_R in order to obtain a closed curve (we speak of *closing the contour*), we would be able to apply either the Cauchy integral theorem – if the closed curve did not contain either of the poles – or the residue theorem – otherwise – in order to evaluate the integral over the full closed contour. If, additionally, it were possible somehow to compute the integral over the additional contour, at least in the limit of large R, we would then be able to compute our original integral.

In general there are multiple ways of closing the contour; i.e., there are multiple different possible choices for the additional curve to be used to produce a closed contour from the original one. Consider the semicircle C_R in the upper half-plane as in the following picture.



Remember that our technique will only be useful if we have a way of computing $\int_{C_R} 1/(1+z^2)\,dz$; we claim that in fact

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{1+z^2}\,dz = 0.$$

In class we showed this by parameterising the curve C_R and considering the resulting integrand; here we use a slightly simpler method. Recall that, if f is continuous on a simple closed curve γ, with maximum M on γ, then

$$\left|\int_\gamma f(z)\,dz\right| \le M|\gamma|,$$

where $|\gamma|$ is the length of the curve γ. Now clearly $|C_R| = \pi R$ (since C_R is a *semi*circle of radius R); further, if z is any point on C_R then we may write $z = Re^{it}$ for some $t \in [0, \pi]$, and thus

$$\left|\frac{1}{1+z^2}\right| = \left|\frac{1}{1+R^2e^{2it}}\right| = \left|\frac{R^{-2}}{e^{2it}+R^{-2}}\right| = R^{-2}\left|e^{2it}+R^{-2}\right|^{-1}.$$

Now by the triangle inequality we may write

$$\left|e^{2it}+R^{-2}\right| \ge \left|e^{2it}\right| - \left|R^{-2}\right| = 1 - R^{-2},$$

so that when we take the limit $R\to\infty$ we have

$$R^{-2}\left|e^{2it}+R^{-2}\right|^{-1} \le R^{-2}(1-R^{-2})^{-1} \le \frac{1}{R^2-1},$$

which goes to zero. Thus

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{1+z^2}\,dz = 0$$

as claimed.

Now for any $R > 1$, the closed curve $L_R + C_R$ will enclose the single pole at i. Let us calculate the residue of $1/(1+z^2)$ at $z = i$. We have

$$\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)} = \frac{1/(z+i)}{z-i};$$

if we think of expanding $1/(z+i)$ as

$$\frac{1}{z+i} = \sum_{n=0}^{\infty} c_n(z-i)^n$$

(which we can do since $1/(z+i)$ is analytic near i), then

$$\frac{1}{1+z^2} = \frac{c_0 + c_1(z-i) + c_2(z-i)^2 + \cdots}{z-i} = \frac{c_0}{z-i} + c_1 + c_2(z-i) + \cdots,$$

and the residue of $1/(1+z^2)$ is clearly c_0. But $c_0 = 1/(z+i)|_{z=i} = 1/(2i)$. Thus at the end of the day we have for all $R > 1$

$$\int_{L_R}\frac{1}{1+z^2}\,dz + \int_{C_R}\frac{1}{1+z^2}\,dz = 2\pi i\cdot\frac{1}{2i} = \pi;$$

and in the limit $R\to\infty$, the integral over L_R goes to $\int_{-\infty}^{+\infty} 1/(1+x^2)\,dx$ while the integral over C_R goes to zero. Thus at the end of the day

$$\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\,dx = \pi,$$

exactly as we found above.

There are two key points in the above procedure: (i) we have to find a curve C_R (which need not be a semicircle, in general, or any segment of a circular path) which will close the contour L_R, and over which we can integrate f; (ii) we have to evaluate the residues of f at its singularities inside the closed contour $L_R + C_R$. We shall now give methods for addressing these two points: first, by providing general conditions under which integrals along circular arcs like C_R go to zero as $R \to \infty$; second, by providing additional methods for calculating residues.

29. When $\int_{C_R} f(z)\,dz \to 0$. First we have the following fairly straightforward generalisation of the example from the previous section. Suppose that f is a function analytic on the exterior of a circle of radius R for a suitably large R (in other words, if f has any singularities they are not too far from the origin). Suppose that for suitably large R there is a number M_R such that for all $z \in C_R$ we have $|f(z)| < M_R$, and that $\lim_{R\to\infty} RM_R = 0$. Then

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$$

To see this, note that the length of C_R is πR; thus

$$\left| \int_{C_R} f(z)\,dz \right| \leq \pi R M_R,$$

and as this latter quantity goes to zero by assumption, the integral must also, by the squeeze theorem.

We may apply this to the example in the previous section as follows. Suppose that $z \in C_R$. Then we have

$$|f(z)| = \left| \frac{1}{1+z^2} \right| \geq \frac{1}{|z^2| - 1} = \frac{1}{R^2 - 1},$$

where we have used the triangle inequality as before; since $R/(R^2 - 1) \to 0$ as $R \to \infty$, the above result shows that $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$ as well.

We note in passing that it is actually sufficient to show that if $z \to \infty$ along circles C_R, then we must have $\lim_{z\to\infty} zf(z) = 0$. More precisely, what this means is that $zf(z)$ can be made arbitrarily small by taking $z \in C_R$ with $|z| = R$ arbitrarily large. In cases like the foregoing this is easier to apply, since we have

$$\lim_{z\to\infty} \frac{z}{1+z^2} = \lim_{z\to\infty} \frac{z^{-1}}{1+z^{-2}},$$

and since the numerator goes to 0 while the denominator goes to 1, the fraction must go to zero. To be fully rigorous, though, we would have to explain how this kind of a limit – restricting z to lie on a particular family of curves – relates to usual limits, but we shall not do that here. This method can also be used to show easily what we saw in class: suppose that

$$f(z) = \frac{P(z)}{Q(z)},$$

where P and Q are polynomials, and $\deg Q \geq \deg P + 2$. Then we can write

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z(a_0 + \cdots + a_n z^n)}{b_0 + \cdots + b_{n+2} z^{n+2}},$$

where $b_{n+2} \neq 0$ but we may have $a_n = 0$. By dividing numerator and denominator by z^{n+2}, this becomes

$$\lim_{z\to\infty} \frac{a_n z^{-1} + a_{n-1} z^{-2} + \cdots + a_0 z^{-n-1}}{b_{n+2} + b_{n+1} z^{-1} + \cdots + b_0 z^{-n-2}} = 0,$$

since the numerator goes to 0 while the denominator goes to $b_{n+2} \neq 0$.

Let us do an example.

EXAMPLE. Compute

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx.$$

We note first that the integrand has poles in the complex plane at $\pm i$, just like the example above. Let us now consider what kind of contour we can use to close the line L_R. Now we have

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right).$$

Now suppose that $z = a + ib$; then

$$\cos z = \frac{1}{2}\left(e^{ia-b} + e^{-ia+b}\right),$$

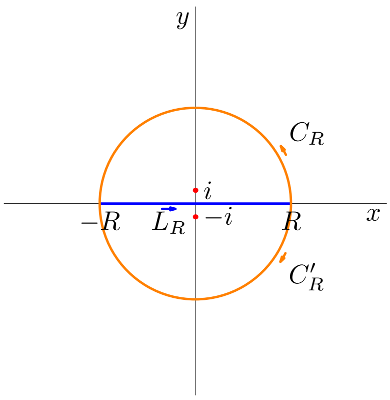
and we see that the first term goes to zero on the upper half-plane ($b > 0$) while the second term goes to infinity exponentially there, and vice versa on the lower half-plane. (We ignore for the moment what happens on the real axis when $b = 0$.) Thus it does not seem that there is any way of closing the contour so as to have $\int_{C_R} f(z)\,dz = 0$ as regardless of whether C_R is in the upper or lower half-plane the integrand will have one term going to infinity.

There are two ways of dealing with this. The more general one is to split the original integral up into two pieces,

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{2(1+x^2)}\,dx, \qquad \int_{-\infty}^{+\infty} \frac{e^{-ix}}{2(1+x^2)}\,dx,$$

and then closing these two integrals in the upper and lower half-plane, respectively; for example, using in turn the curves C_R and C'_R in the following figure. We shall see similar cases to this in the future. For now we use a simpler method. Note that we have also

$$\cos z = \operatorname{Re} e^{iz},$$



so since we are integrating along the real axis, we may write

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx = \int_{-\infty}^{+\infty} \frac{\operatorname{Re} e^{ix}}{1+x^2}\,dx = \operatorname{Re} \int_{-\infty}^{+\infty} \frac{e^{ix}}{1+x^2}\,dx.$$

We try closing this integral in the upper half-plane as described above; thus let C_R denote a semicircle from R to $-R$ in the upper half-plane, as indicated in the above figure. If $z = a + ib \in C_R$, then $b \geq 0$, so we have

$$\left|e^{iz}\right| = \left|e^{ia}e^{-b}\right| = e^{-b} \leq 1,$$

and

$$\left|\frac{e^{iz}}{1+z^2}\right| \leq \frac{1}{|1+z^2|} \leq \frac{1}{R^2-1}$$

as before, and since $\lim_{R\to\infty} R/(R^2-1) = 0$ we have

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{1+z^2}\,dz = 0.$$

Thus we need only calculate the residues of $e^{iz}/(1+z^2)$ in the upper half-plane. Now in the upper half-plane this function is singular only at $z = i$; if we proceed in the same way we did in the previous example (we shall give a general method for this right after this example), we see that the residue is

$$\frac{e^{i\cdot i}}{2i} = \frac{1}{2ei},$$

and finally by the residue theorem and the fact that $\int_{C_R} \frac{e^{iz}}{1+z^2}\,dz \to 0$ as $R \to \infty$, that our original integral is

$$\int_{-\infty}^{+\infty} \frac{e^{iz}}{1+z^2}\,dz = 2\pi i \cdot \frac{1}{2ei} = \frac{\pi}{e}.$$

This is already real, so that we have

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{e}.$$

30. Methods for computing residues. In the previous two examples we have tacitly applied the following result:

Suppose that $f(z)$ has a simple pole at $z = a$. Then

$$\mathrm{Res}_a f(z) = \lim_{z\to a}(z-a)f(z).$$

This is quite simple to see. Since f has a simple pole, there must be a function $\phi(z)$ which is analytic and nonzero at a such that

$$f(z) = \frac{\phi(z)}{z-a}.$$

Then, proceeding as in the two examples above, it is easy to see that $\mathrm{Res}_a f(z) = \phi(a)$; alternatively, we may use the Cauchy integral formula (here γ is a small circle around a such that f is analytic on and within γ):

$$\mathrm{Res}_a f(z) = \frac{1}{2\pi i}\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\int_\gamma \frac{\phi(z)}{z-a}\,dz = \phi(a).$$

But

$$\phi(a) = \lim_{z\to a}\phi(z) = \lim_{z\to a}(z-a)f(z),$$

which establishes our result.

We may extend the above result to poles of higher order. Suppose that f has instead a pole of order m at $z = a$. Then we may write

$$f(z) = \frac{\phi(z)}{(z-a)^m},$$

where as before ϕ is analytic and nonzero at a; thus (letting as before γ denote a small circle around a within and on which f is analytic)

$$\begin{aligned}\mathrm{Res}_a f(z) = \frac{1}{2\pi i}\int_\gamma \frac{\phi(z)}{(z-a)^m}\,dz &= \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\phi(z)\Big|_{z=a}\\ &= \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z).\end{aligned}$$

In other words, to calculate the residue of a function f at a pole of order m, we first multiply f by $(z-a)^m$, differentiate $m-1$ times, evaluate at a, and divide by $(m-1)!$. (Note that $m \geq 1$, so that $m-1 \geq 0$ and the foregoing makes sense.) Note that this formula reduces to the previous one in the case $m = 1$. Note also that to apply it we must first determine the order of the pole a.

In the case that $f(z) = P(z)/Q(z)$, as before, where P and Q have no common factors and Q has no repeated roots, we see that every pole of f will be simple, and the residue at a pole z_0 will be

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \frac{P(z)}{Q(z)/(z - z_0)} = \frac{P(z_0)}{Q'(z_0)},$$

since $Q(z_0) = 0$ (as z_0 is a pole of f and therefore must be a zero of Q) and this allows us to write

$$\lim_{z \to z_0} \frac{Q(z)}{z - z_0} = \lim_{z \to 0} \frac{Q(z) - Q(z_0)}{z - z_0} = Q'(z_0).$$

$Q'(z_0) \neq 0$ since by assumption the roots of Q are not repeated.

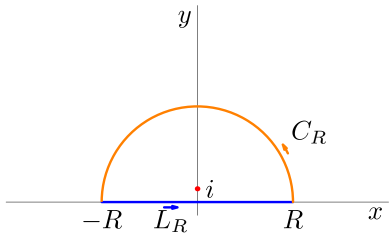
Let us give an example.

EXAMPLES. 1. Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{1}{(1 + x^2)^2} \, dx.$$

We note that the integrand, extended to the complex plane, has poles at $\pm i$, each of order 2. We expect that we can close the contour using a half-circle C_R from R to $-R$ in the upper half-plane, as we have done in the other examples above (see the picture). To see that we can in fact do this, we apply the result from the previous section: for $z \in C_R$, we have

$$|1 + z^2| \geq R^2 - 1, \qquad |1 + z^2|^2 \geq (R^2 - 1)^2, \qquad \left| \frac{1}{(1 + z^2)^2} \right| \leq \frac{1}{(R^2 - 1)^2},$$



and since $R/(R^2 - 1)^2$ clearly goes to zero as $R \to \infty$, we have

$$\lim_{R \to \infty} \int_{C_R} \frac{1}{(1 + z^2)^2} \, dz = 0.$$

Thus we need only calculate the residue of $1/(1 + z^2)^2$ at i. Since i is a pole of order 2 of $1/(1 + z^2)^2$, this will be, since $(1 + z^2)^2 = (z - i)^2 (z + i)^2$,

$$\frac{1}{(2 - 1)!} \lim_{z \to i} \frac{d^{2-1}}{dz^{2-1}} (z - i)^2 \cdot \frac{1}{(1 + z^2)^2} = \frac{d}{dz} \frac{1}{(z + i)^2} \bigg|_{z=i} = -\frac{2}{(z + i)^3} \bigg|_{z=i} = -\frac{2}{-8i} = -\frac{1}{4} i,$$

and the integral will be

$$\int_{-\infty}^{+\infty} \frac{1}{(1 + x^2)^2} \, dx = 2\pi i \cdot \left(-\frac{1}{4} i \right) = \frac{\pi}{2}.$$

(It is worth pointing out here that it is *always* a good idea to make sure that our final answer makes sense: for example, here we are integrating a real-valued function, so we expect to get a real number as the result; and it is in fact a *positive* real-valued function, so we expect to get a positive real number as the result, as we have. Had we gotten a negative real number, or a complex number with a nonzero imaginary part, it would mean we had made a mistake somewhere earlier which we would need to go back and fix.)

2. Let us evaluate the integral

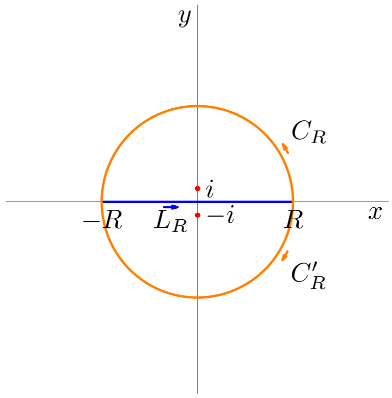
$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1 + x^2)^2} \, dx.$$

Here we clearly have the same kind of issue that we had in our example in section 29 above; namely, whether we close the contour in the upper or the lower half-plane will depend on the sign of k. Let us first suppose that $k \geq 0$. Then if $z = a + ib$ is in the upper half-plane, so that $b \geq 0$, then as in the example just cited we have

$$e^{ikz} = e^{-kb}e^{ika},$$

which is bounded in absolute value by $e^{-kb} \leq 1$; in other words, the integrand here on the semicircle C_R in the next figure will be bounded by the same quantity as we had for $1/(1+z^2)^2$ in the previous example, and the integral over C_R will go to zero as $R \to \infty$ as there. More carefully, recall that we just showed that on C_R

$$\left|\frac{1}{(1+z^2)^2}\right| \leq \frac{1}{(R^2-1)^2};$$



thus when $k \geq 0$ and $z \in C_R$ (so that z is in the upper half-plane)

$$\left|\frac{e^{ikz}}{(1+z^2)^2}\right| \leq \frac{1}{(R^2-1)^2}$$

as well, and since $R/(R^2-1)^2 \to 0$ as $R \to \infty$, we have

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{ikz}}{(1+z^2)^2}\, dz = 0$$

by our general results above. Thus it suffices to calculate the residue of $e^{ikz}/(1+z^2)^2$ at the pole i, and since this is still a pole of order 2, its residue is

$$\frac{1}{(2-1)!} \lim_{z\to i} \frac{d^{2-1}}{dz^{2-1}}(z-i)^2 \cdot \frac{e^{ikz}}{(1+z^2)^2} = \frac{d}{dz}\frac{e^{ikz}}{(z+i)^2}\bigg|_{z=i}$$
$$= \frac{ike^{ikz}}{(z+i)^2}\bigg|_{z=i} - 2\,\frac{e^{ikz}}{(z+i)^3}\bigg|_{z=i} = \frac{ike^{-k}}{-4} - 2\frac{e^{-k}}{-8i} = -ie^{-k}\left(\frac{1}{4} + \frac{1}{4}k\right),$$

so that the integral for $k \geq 0$ is

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1+x^2)^2}\, dx = 2\pi i \cdot (-i)e^{-k}\left(\frac{1}{4} + \frac{1}{4}k\right) = \frac{\pi}{2}e^{-k}\,(k+1)\,.$$

Now suppose that $k \leq 0$, and consider the curve C_R' in the above figure. If $z = a + bi \in C_R'$, then $b \leq 0$, so

$$e^{ikz} = e^{-kb}e^{ika}$$

will still be bounded by 1 in absolute value since $k \leq 0$ and $b \leq 0$ implies that $kb \geq 0$, i.e., $-kb \leq 0$. The exact same logic used above now shows that

$$\lim_{R\to\infty} \int_{C_R'} \frac{e^{ikz}}{(1+z^2)^2}\, dz = 0,$$

and we are left with calculating the residue at $-i$. This is very similar to calculating the residue at i; it is equal to

$$\frac{d}{dz}\frac{e^{ikz}}{(z-i)^2}\bigg|_{z=-i} = \frac{ike^{ikz}}{(z-i)^2}\bigg|_{z=-i} - 2\,\frac{e^{ikz}}{(z-i)^3}\bigg|_{z=-i} = \frac{ike^k}{-4} - 2\frac{e^k}{8i} = ie^k\left(\frac{1}{4} - \frac{1}{4}k\right).$$

Before we can determine the value of the integral, though, there is one additional wrinkle we have not yet mentioned: note that the curve $L_R + C_R$ was oriented counterclockwise, as required by the Cauchy integral formula; but $L_R + C_R'$ is oriented *clockwise*, which means that we must put in an extra minus sign when applying the Cauchy integral formula. More carefully, we have

$$\int_{L_R} \frac{e^{ikz}}{(1+z^2)^2}\,dz + \int_{C_R'} \frac{e^{ikz}}{(1+z^2)^2}\,dz = -2\pi i \mathrm{Res}_{-i}\frac{e^{ikz}}{(1+z^2)^2};$$

thus, finally, our integral is

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1+x^2)^2}\,dx = -2\pi i\cdot\left[ie^k\left(\frac{1}{4} - \frac{1}{4}k\right)\right] = \frac{\pi}{2}e^k\,(1-k)\,.$$

Pulling all of this together, then, we have finally that

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1+x^2)^2}\,dx = \frac{\pi}{2}e^{-|k|}(1+|k|),$$

since $|k| = k$ when $k \geq 0$ and $|k| = -k$ when $k \leq 0$.

For those who know something about Fourier transforms, it is interesting to note the following about the differentiability of this function. If we expand the exponential out in a Taylor series, we see that (dropping the $\pi/2$ coefficient for convenience)

$$\begin{aligned} e^{-|k|}(1+|k|) &= \left(1 - |k| + \frac{1}{2}|k|^2 - \frac{1}{6}|k|^3 + \cdots\right)(1+|k|) \\ &= 1 - |k|^2 + \frac{1}{2}|k|^2 + \frac{1}{2}|k|^3 - \frac{1}{6}|k|^3 + \cdots. \end{aligned}$$

Now a little thought should convince you that $|k|^n$ has n derivatives everywhere and $n+1$ derivatives everywhere except 0, where the $n+1$th derivative is discontinuous. This shows that $e^{-|k|}(1+|k|)$ has 3 continuous derivatives, which agrees nicely with the fact that the function $1/(1+x^2)^2$ has 3 moments in L^2 (i.e., that the integral of the square of $x^n/(1+x^2)^2$ over all of $\mathbf{R}^1$ will be finite for $n \leq 3$).

Summary:

- We tie down a few loose ends from previous lectures.
- We prove Jordan's lemma and give examples of its application to the evaluation of definite integrals.
- We then give additional examples of finding contours in the complex plane for the evaluation of definite integrals on the real line.

(Goursat, §§41, 44 – 46.)

31. On zeroes and poles. Recall that we have defined poles of a definite order and zeroes of a definite order, as follows:

Pole of order m at a:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{m} b_n (z-a)^{-n}, \quad b_m \neq 0$$
$$= \frac{\phi(z)}{(z-a)^m}, \quad \phi \text{ analytic and nonzero at } a$$

Zero of order m at a:

$$f(z) = \sum_{n=m}^{\infty} a_n (z_a)^n, \quad a_m \neq 0$$
$$= (z-a)^m \phi(z), \quad \phi \text{ analytic at } a.$$

We now claim that function which is analytic except for isolated singularities can have only finitely many poles, and that a nonzero analytic function can have only finitely many zeroes, on any finite region. Thus, let C be a simple closed curve on and within which the function f is analytic, and suppose first that f has infinitely many zeroes at a_1, a_2, $\cdots$ within the curve C. We shall only give the main idea (the details will be given later when we talk about analytic continuation). We need the celebrated *Bolzano-Weierstrass Theorem*:

Let $\{a_1, a_2, \cdots\}$ be an infinite set within a simple closed curve C. Then there must be a point a within or on C such that every disk around a contains infinitely many points of this set.

This is proved in courses on analysis, but it is also quite reasonable intuitively since if there infinitely many points in a finite region, surely they cannot all be staying a finite distance away from each other: they must be 'clustering' somewhere.[1] By this theorem, there must be a point a within or on C such that every disk around a contains infinitely many zeroes of f; thus any disk around a must contain some point at which f is zero, which means that $f(a)$ must be zero: in other words, a is a zero of f. Let us write out the Taylor series of f at a:

$$f(z) = \sum_{n=1}^{\infty} \alpha_n (z-a)^n.$$

We claim that all of the coefficients must be zero. Suppose that $\alpha_k \neq 0$ for some k. Then we would be able to write

$$f(z) = (z-a)^k \phi(z),$$

where ϕ is analytic at a and – crucially – *nonzero* at a. Now this would imply that there would be a disk around a on which ϕ is still nonzero; but since $(z-a)^k$ is zero only when $z = a$, f would not be zero anywhere on this disk either, contradicting our choice of a. Thus all of the coefficients in the Taylor series of f must be zero, which means that f must be identically zero on every disk around a at which it is analytic. Note though that this does not automatically allow us to conclude that it must be identically zero on C. The idea to complete the proof – which we shall go over more carefully when we talk about analytic continuation later – is as follows. f must be analytic on some disk around a. Now let us take a point near the boundary of this disk; then since f is identically zero near that point, its Taylor series around that point must still be identically zero. Thus f must be identically zero on all disks around this new point on which it is still analytic. We can then continue extending the region until we show that f must actually be identically zero on all of C. (Specifically, as we shall see when we talk about analytic continuation, we actually proceed by extending f along a curve to any other point in C, which allows us to conclude that f must still be zero at that point, and hence at every point in C.)

[1] In other words, there is no way for an infinite group of people to practice social distancing within a grocery store!

Now suppose that the points a_1, a_2, $\cdots$ were in fact poles. Then as before there would be a point a, any disk around which would contain infinitely many of the poles a_i. Then clearly a cannot be an isolated singularity of f, since any disk around it contains additional singularities of f; but a cannot be a regular point either, since f is not analytic on any disk around a. Thus f could not be analytic except for isolated singularities, completing the proof in this case.

32. Cauchy principal value. Recall that in elementary calculus we give the following definition:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{L_1\to-\infty}\int_{L_1}^{a} f(x)\,dx + \lim_{L_2\to\infty}\int_{a}^{L_2} f(x)\,dx, \tag{1}$$

where the integral on the left exists if and only if the two limits on the right-hand side both exist as finite numbers. Here a is any real number; it is easy to show that the definition does not depend on the choice of a, so for convenience we shall take $a = 0$.

Now the careful student may have noted that the integrals we have calculated so far are *not* in the form of a sum of two different limits, but rather of a single limit,

$$\lim_{L\to\infty}\int_{-L}^{L} f(x)\,dx.$$

This limit is called the *Cauchy principal value* of the integral, and we denote it by $\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx$.[2] Now it is easy to see that if the integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists as defined above, then the Cauchy principal value also exists and is equal to it; for in this case

$$\begin{aligned}\lim_{L\to\infty}\int_{-L}^{L} f(x)\,dx &= \lim_{L\to\infty}\int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx\\ &= \lim_{L\to\infty}\int_{-L}^{0} f(x)\,dx + \lim_{L\to\infty}\int_{0}^{L} f(x)\,dx = \lim_{L_1\to-\infty}\int_{L_1}^{0} f(x)\,dx + \lim_{L_2\to\infty}\int_{0}^{L_2} f(x)\,dx\\ &= \int_{-\infty}^{\infty} f(x)\,dx.\end{aligned}$$

The logic, however, does not work in reverse, and for a very simple reason. Note that in going from the first to the second line above we used the fact that if the limit of two quantities exist, then the limit of their sum exists and equals the sum of the limits. It is, however, most definitely *not* true that if the limit of a sum exists, then the limit of the two terms in the sum both exist! (As a simple example, consider $f(x) = 1 - 1/x$ and $g(x) = 1/x$ as $x \to 0$: clearly, $f(x) + g(x) = 1$, and the limit of this exists as $x \to 0$, while neither f nor g has a limit which exists.) Thus the logic cannot be run backwards. To sum up, then: if $\int_{-\infty}^{\infty} f(x)\,dx$ exists, so does $\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx$, and the two must be equal; but the converse is not necessarily true.

There is one case where the converse is true, though: when f is even. In this case, we see that

$$\lim_{L\to\infty}\int_{-L}^{L} f(x)\,dx = 2\lim_{L\to\infty}\int_{0}^{L} f(x)\,dx,\ \lim_{L\to-\infty}\int_{L}^{\infty} f(x)\,dx = \lim_{L\to\infty}\int_{0}^{L} f(x)\,dx,$$

so that if the Cauchy principal value exists, then so do both of the limits in (1) above, and hence so does the integral $\int_{-\infty}^{\infty} f(x)\,dx$. To summarise, then, we have

$$\text{if the integral } \int_{-\infty}^{\infty} f(x)\,dx \text{ exists, then so does PV} \int_{-\infty}^{\infty} f(x)\,dx, \text{ and the two are equal;}$$

$$\text{if PV} \int_{-\infty}^{\infty} f(x)\,dx \text{ exists and } f \text{ is even, then so does } \int_{-\infty}^{\infty} f(x)\,dx \text{ and the two are equal.}$$

[2] The terminology may be slightly misleading. The Cauchy principal value is something which can be computed independently of the actual integral as defined above: it requires only evaluating the single limit just given. In particular, we do *not* determine the Cauchy principal value by first evaluating the full integral and then doing something to that number!

Another way of looking at the difference between (1) and the principal value is to note that in (1) we have a two-dimensional limit, while the principal value is effectively the directional limit along the line $L_1 = -L_2$. As we learned in multivariable calculus, if the two-dimensional limit of a quantity exists, then so does the limit along any curve – but if all we know is that the limit along one particular line exists, we really do not know anything at all about the full two-dimensional limit, in general. Thus knowing that the Cauchy principal value exists does not, in general, tell us anything about the integral in (1).

It is worth noting that the techniques we have studied so far all amount to calculating the Cauchy principal value rather than the integral as defined in (1). Hence, if we are asked to compute the integral $\int_{-\infty}^{\infty} f(x)\,dx$, in order to show that it equals the Cauchy principal value we must first show that it exists (as it will, as just shown, when f is even, for example).

33. Jordan's lemma. Recall that there are two main steps to computing integrals using contours: one, finding a way of 'closing the contour' in such a way that we can calculate the integral of our function over the additional part of the contour (for example, using a semicircle the integral over which goes to zero as its radius goes to infinity); two, evaluating residues. In the previous lecture we saw additional methods for the second step; now we shall prove a result helping us to deal with the first step. First of all, we note that for $x \in [0, \pi/2)$ we have $0 \le \cos x \le 1$, so

$$\frac{d}{dx}\frac{\sin x}{x} = \frac{x\cos x - \sin x}{x^2} = \frac{x - \tan x}{x^2}\cos x \le 0,$$

since $\tan x \ge x$ for $x \in [0, \pi/2)$. This means that the function $\sin x/x$ is decreasing on $[0, \pi/2)$, so its minimum value on $[0, \pi/2]$ is achieved at $x = \pi/2$, and is therefore $(\sin \pi/2)/(\pi/2) = 2/\pi$. Thus for $x \in [0, \pi/2]$ we have $\sin x \ge \frac{2}{\pi}x$.

With this preliminary, we may now prove Jordan's lemma:

Let C_R denote the semicircle of radius R centred at the origin in the upper half-plane. Let f be a function which is analytic in the upper half-plane on the exterior of some semicircle of radius R_0, and such that for every $R > R_0$ there is a constant M_R such that $|f(z)| \le M_R$ on C_R, and $M_R \to 0$ as $R \to \infty$. Then $\int_{C_R} f(z)e^{iaz}\,dz \to 0$ as $R \to \infty$ for any positive number a.

To prove this, parameterise C_R by $z(t) = R(\cos t + i\sin t)$, $t \in [0, \pi]$; then

$$e^{iaz} = e^{iaR(\cos t + i\sin t)} = e^{iaR\cos t}e^{-aR\sin t}.$$

Now the first factor has modulus one, while for $t \in [0, \pi/2]$ we have

$$\sin t \ge \frac{2}{\pi}t, \qquad -\sin t \le -\frac{2}{\pi}t, \qquad e^{-aR\sin t} \le e^{-\frac{2aR}{\pi}t};$$

thus

$$\begin{aligned}\left|\int_{C_R} f(z)e^{iaz}\,dz\right| &\le \int_0^\pi \left|f(Re^{it})\right| e^{-aR\sin t}R\,dt \le RM_R\int_0^\pi e^{-aR\sin t}\,dt = 2RM_R\int_0^{\pi/2} e^{-aR\sin t}\,dt \\ &\le 2RM_R\int_0^{\pi/2} e^{-\frac{2aR}{\pi}t}\,dt = -\frac{M_R\pi}{a}e^{-\frac{2aR}{\pi}t}\Big|_0^{\pi/2} = \frac{M_R\pi}{a}\left(1 - e^{-aR}\right),\end{aligned}$$

which goes to zero as $R \to \infty$, since M_R does and the quantity in parentheses goes to 1. This completes the proof.

We note that the same is true if we replace 'upper' everywhere by 'lower' and require that a be negative: for now we may parameterise C_R by $z(t) = -R(\cos t + i\sin t)$, in the which case, proceeding as before, the integral over C_R of $f(z)e^{iaz}$ can be bounded by

$$2RM_R\int_0^{\pi/2} e^{aR\sin t}\,dt.$$

But now, as before, aR is negative since a is, and the proof proceeds as before with $-aR$ replaced by aR everywhere.

We now give some examples.

EXAMPLES. 1. Evaluate $\int_0^\infty \frac{x \sin x}{1+x^2}\, dx$.

We first note that the integrand is even so that we have

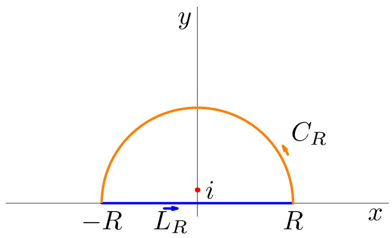
$$\int_0^\infty \frac{x \sin x}{1+x^2}\, dx = 2\int_{-\infty}^\infty \frac{x \sin x}{1+x^2}\, dx,$$

where this last integral will exist exactly when the principal value exists, as shown above. Thus it suffices to compute the principal value of this last integral. We wish to apply Jordan's lemma. Now we have $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$; but if we were to use this formula, it would require us to compute two separate integrals which would need to be closed in different half-planes. That would be possible but would be more work than is necessary. Instead we write

$$\sin x = \operatorname{Im} e^{ix},$$

and note that this allows us to write

$$\int_{-\infty}^\infty \frac{x \sin x}{1+x^2}\, dx = \operatorname{Im} \int_{-\infty}^\infty \frac{xe^{ix}}{1+x^2}\, dx.$$



This looks like the kind of function to which we should be able to apply Jordan's lemma. We only need a bound on $x/(1+x^2)$ on the upper semicircle C_R. Now on C_R we have

$$\left|\frac{z}{1+z^2}\right| = \frac{|z|}{|1+z^2|} \geq \frac{R}{R^2-1} = \frac{1/R}{1-R^{-2}},$$

which clearly goes to zero as $R \to \infty$. Thus by Jordan's lemma

$$\int_{C_R} \frac{ze^{iz}}{1+z^2}\, dz \to 0 \quad \text{as} \quad R_{\to}\infty.$$

Since the integrand has only one pole in the upper half-plane, at $z = i$, we may write, by the residue theorem,

$$\begin{aligned} \int_{-\infty}^\infty \frac{xe^{ix}}{1+x^2}\, dx &= \lim_{R\to\infty}\left[-\int_{C_R} \frac{ze^{iz}}{1+z^2}\, dz + 2\pi i \operatorname{Res}_i \frac{ze^{iz}}{z^2+1}\right] \\ &= 2\pi i \operatorname{Res}_i \frac{ze^{iz}}{(z-i)(z+i)} = 2\pi i \frac{ie^{i^2}}{2i} = \frac{2\pi i}{e}. \end{aligned} \tag{2}$$

Thus we have finally

$$\int_0^\infty \frac{x \sin x}{1+x^2}\, dx = \frac{1}{2}\operatorname{Im} \int_{-\infty}^\infty \frac{xe^{ix}}{1+x^2}\, dx = \frac{1}{2}\operatorname{Im}\frac{2\pi i}{e} = \frac{\pi}{e}.$$

It is worth noting that the integral in (2) is a pure imaginary number; this can be traced to the fact that $x \cos x$ is odd, which means that the principal value of its integral over the real line is zero.

2. Evaluate

$$\int_{-\infty}^\infty \frac{\sin x}{x}\, dx. \tag{3}$$

This integral introduces some additional twists to our standard procedure. First of all, by $\sin x/x$ we mean actually the function

$$\begin{cases} \frac{\sin z}{z}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

which by what we have seen on a previous homework assignment is analytic everywhere on the complex plane, and in fact has the Taylor series expansion

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}.$$

If we now think of closing the integral in (3) in the upper half-plane, it would appear that it evaluates to zero since there are no singularities and hence no residue. This would be wrong (and looking at a graph of $\sin x/x$ suggests as much), since as we have noticed before the integral over a semicircle in the upper half-plane of something involving $\sin x$ will not, in general, go to zero since $\sin x$ includes a term e^{-ix}. Let us look at this a bit more carefully. Let L_R, as usual, denote the line segment from $-R$ to R along the real axis, and C_R denote the upper semicircle of radius R centred at the origin. Then we have by the Cauchy integral theorem

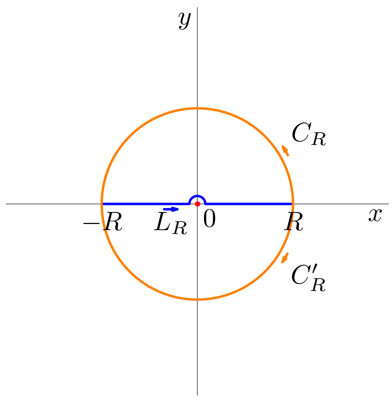
$$0 = \int_{L_R} \frac{\sin x}{x}\, dx + \int_{C_R} \frac{\sin z}{z}\, dz = \int_{L_R} \frac{\sin x}{x}\, dx + \int_{C_R} \frac{e^{iz} - e^{-iz}}{2iz}\, dz.$$

We would like to break this up in such a way that we can close the integral involving e^{-iz} in the lower half-plane. This suggests splitting $\sin x$ up in the first integrand:

$$\int_{L_R} \frac{\sin x}{x}\, dx = \int_{L_R} \frac{e^{ix} - e^{-ix}}{2ix}\, dx.$$

This is perfectly fine, but unfortunately we cannot break this integral up into two separate pieces as it stands since the individual pieces would have a singularity at the origin, which lies on the line L_R. (Note that we do need to break this integral up in order to obtain a closed curve with either C_R or $-C_R$ – the lower semicircle – and hence to apply the residue theorem.) But by the Cauchy integral theorem, since $\sin z/z$ is analytic everywhere on the plane, we may replace L_R with any other curve passing from $-R$ to R; let us use a contour which goes along the real axis from $-R$ to $-\epsilon$ and ϵ to R, and joins $-\epsilon$ to ϵ by a small semicircle of radius ϵ centred at the origin, in the upper half-plane. Denote this contour by L'_R. Then we have

$$\int_{L_R} \frac{\sin x}{x}\, dx = \int_{L'_R} \frac{\sin z}{z}\, dz = \int_{L'_R} \frac{e^{iz}}{2iz}\, dz - \int_{L'_R} \frac{e^{-iz}}{2iz}\, dz.$$



We may now evaluate these integrals by closing in the upper and lower half-planes, respectively. Let us look at the first integral. By the residue theorem,

$$\int_{L'_R} \frac{e^{iz}}{2iz}\, dz + \int_{C_R} \frac{e^{iz}}{2iz}\, dz = 0.$$

But now on C_R we clearly have

$$\left|\frac{1}{2iz}\right| = \frac{1}{2}R^{-1} \to 0 \quad \text{as} \quad R \to \infty,$$

so by Jordan's lemma we have

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{2iz}\, dz = 0.$$

Thus

$$\lim_{R\to\infty} \int_{L'_R} \frac{e^{iz}}{2iz}\, dz = 0.$$

The second integral is more interesting. We have by the residue theorem

$$\int_{L'_R} \frac{e^{-iz}}{2iz}\, dz + \int_{-C_R} \frac{e^{-iz}}{2iz}\, dz = 2\pi i \mathrm{Res}_0 \frac{e^{-iz}}{2iz} = \pi,$$

while $|1/(2iz)| = 1/(2R) \to 0$ as $R \to \infty$ shows that the second integral vanishes as $R \to \infty$, by Jordan's lemma. Thus we have

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \mathrm{PV} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \lim_{R\to\infty} \int_{L_R} \frac{\sin x}{x}\, dx = \pi.$$

It is worth noting that the method we used in the previous example – of replacing $\sin x$ by e^{ix} and then taking an imaginary part – does not work directly in this case since the curve L'_R we integrate over does not lie along the real axis, so we cannot simply recover the integral over it of $\sin x/x$ from that of e^{ix}/x by taking an imaginary part.

34. Another way of closing the contour. Let us consider, by way of example, another method for closing the contour.

Example. Evaluate the integrals

$$\int_0^{\infty} \sin x^2\, dx, \qquad \int_0^{\infty} \cos x^2\, dx.$$

We do this by considering the integral

$$\int_0^{\infty} e^{iz^2}\, dz.$$

Let L_R in this case denote the line segment from 0 to R along the real axis. We shall close the contour in two different ways. First, though, it is probably worthwhile to consider why the methods we have been using so far do not work in this case. Clearly, if the above integral exists then it will equal half the Cauchy principal value of

$$\int_{-\infty}^{\infty} e^{iz^2}\, dz.$$

Now if we consider the integral from $-R$ to R of e^{iz^2} and then close it with the semicircle C_R of radius R centred at the origin in the upper half-plane, then it would appear initially – as in the previous example – that we would get zero. However, as before, the integral along C_R of e^{iz^2} does not vanish as $R \to \infty$. While not a proof, we may see that this is reasonable by the following computation:

$$\int_{C_R} e^{iz^2}\, dz = \int_0^{\pi} e^{iR^2 e^{2it}} iRe^{it}\, dt = \int_0^{\pi} e^{iR^2 \cos 2t} e^{-R^2 \sin 2t} iRe^{it}\, dt;$$

but $\sin 2t < 0$ when $t \in (\pi/2, \pi)$, so that the exponential above goes to infinity with R on that range of t. Thus evidently we need to do something else.

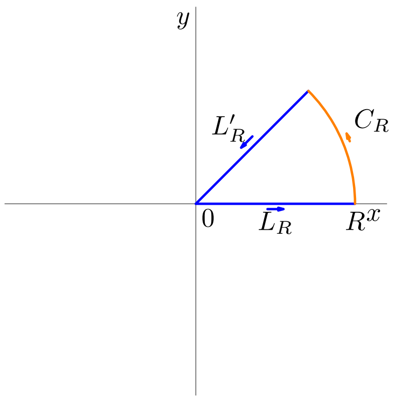
This calculation actually suggests something worth noting: if we were able to restrict $t \in (0, \pi/2)$, then the exponential above would go to zero as $R \to \infty$, and it is possible that the whole integral will also go to zero. Thus let us consider closing the contour with a piece C_R' of the full semicircle C_R together with a line segment back to the origin, i.e., with a pie-wedge shaped contour as in the following figure. The problem now will be how to calculate the integral over the additional line segment L_R'. Now the line segment L_R' may be parameterised as $\omega(R-t)$, $t \in [0, R]$ (the $R - t$ is because the line starts on C_R and ends at the origin), where ω is some complex number of unit modulus. This allows us to write the integral over L_R' as

$$\int_0^R e^{i\omega^2(R-t)^2}\,\omega(-dt) = -\omega\int_0^R e^{i\omega^2t^2}\,dt.$$

Now if $\omega^2 = i$, then the integrand would become e^{-t^2}, and we can compute the integral of e^{-t^2} over the positive real axis by other methods, so it appears that we might be able to use the line L_R' in that case. We shall do this in detail below. Alternatively, if $\omega^2 = -1$, then the integral over L_R' will be $-\omega$ times the conjugate of that over L_R, so we may be able to find the integral over L_R in this case also by isolating and solving. We shall not use this method here, but it is very useful for this week's homework assignment. (Hint, hint!)

We shall thus take ω to satisfy $\omega^2 = i$. This means that we have two choices: $\omega = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ and $\omega = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$. We take the first because to close to the second would require us to use a circle along which we do not have good bounds for e^{iz^2}. Thus we let L_R' be the line parameterised by $(R-t)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$, $t \in [0, R]$, and let C_R' denote the segment of the semicircle of radius R centred at the origin from R to $R\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$. Then by the Cauchy integral theorem we have

$$\int_{L_R} e^{iz^2}\,dz + \int_{C_R'} e^{iz^2}\,dz + \int_{L_R'} e^{iz^2}\,dz = 0.$$



We deal with the integral over C_R' first. We have

$$\left|\int_{C_R'} e^{iz^2}\,dz\right| = \left|\int_0^{\pi/4} e^{iR^2e^{2it}}Rie^{it}\,dt\right| \le R\int_0^{\pi/4} e^{-R^2\sin 2t}\,dt;$$

since $t \in [0, \pi/4]$, we have $2t \in [0, \pi/2]$, so $\sin 2t \ge \frac{2}{\pi}(2t) = \frac{4}{\pi}t$ and the above integral is bounded by

$$R\int_0^{\pi/4} e^{-\frac{4R^2}{\pi}t}\,dt = -\frac{\pi}{4R}e^{-\frac{4R^2}{\pi}t}\Big|_0^{\pi/4} = \frac{\pi}{4R}\left(1 - e^{-R^2}\right),$$

which goes to zero as $R \to \infty$. Thus the integral over C_R' does not contribute anything to the final integral. Now the integral over L_R' is equal to

$$\int_0^R e^{-(R-t)^2}\left[-\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right]dt = -e^{i\pi/4}\int_0^R e^{-t^2}\,dt,$$

which in the limit as $R \to \infty$ becomes

$$-e^{i\pi/4}\frac{1}{2}\sqrt{\pi} = -\frac{1}{2}\sqrt{\frac{\pi}{2}} - i\frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Thus we have finally

$$\int_0^\infty e^{iz^2}\,dz = -\lim_{R\to\infty}\int_{L'_R} e^{iz^2}\,dz = \frac{1}{2}\sqrt{\frac{\pi}{2}} + i\frac{1}{2}\sqrt{\frac{\pi}{2}},$$

so we see that

$$\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Summary:
- We demonstrate additional methods of computing definite integrals using contours.
- We then proceed to discuss more general properties of analytic functions.
- In particular, we prove Liouville's Theorem, the argument principle, and Rouché's Theorem, and use these to give two different proofs of the fundamental theorem of algebra.
- We then discuss the Poisson kernel for Laplace's equation on a disk.

(Goursat, §§36, 45, 48 – 49.)

35. Additional methods of closing the contour. We show two more methods of closing the contour on a definite integral, by way of example.

Example. Evaluate the integral

$$\int_0^\infty \frac{x^{1/3}}{(1+x^2)^2}\,dx. \tag{1}$$

We note first of all that this integral converges (apply the usual power test). Now to extend this to a contour integral over a complex curve we must first choose a branch of the cube root function. We shall choose the branch with a branch cut along the positive real axis*, and take the angle θ to lie in the interval $(0, 2\pi)$. To evaluate integral (1), we consider a so-called *keyhole contour* composed of four separate curves (see the figure): a line L_R from $i\epsilon$ to $R + i\epsilon$; a circular arc C_R running counterclockwise from $R + i\epsilon$ to $R - i\epsilon$, centred at the origin; another line L'_R running from $R - i\epsilon$ to $-i\epsilon$; and finally a semicircle C'_ϵ running clockwise from $-i\epsilon$ to $i\epsilon$ in the third and second quadrants, again centred at the origin. (There are of course other slightly different contours which would perform the same task equally well.) Let us let $\gamma = L_R + C_R + L'_R + C'_\epsilon$ denote the full curve. Note that since the branch point and branch cut of $z^{1/3}$ lie entirely outside of this curve (this is our first indication that taking a branch cut along the line of integration was in fact the correct thing to do!) the integrand $z^{1/3}/(1 + z^2)^2$ has only poles within the contour, at $z = \pm i$, we see from the residue theorem that

$$\int_\gamma \frac{z^{1/3}}{(1+z^2)^2}\,dz = 2\pi i\left[\mathrm{Res}_i \frac{z^{1/3}}{(1+z^2)^2} + \mathrm{Res}_{-i}\frac{z^{1/3}}{(1+z^2)^2}\right].$$

We will come back to the calculation of these residues later and consider first how the integral at left relates to the integral in (1). In particular, we claim that

$$\int_{C_R} \frac{z^{1/3}}{(1+z^2)^2}\,dz \to 0 \quad \text{and} \quad \int_{C'_\epsilon} \frac{z^{1/3}}{(1+z^2)^2}\,dz \to 0 \quad \text{as} \quad R\to\infty,\ \epsilon\to 0^+. \tag{2}$$

Note first that if z is on C_R or C'_ϵ, then $|z^{1/3}| = |z|^{1/3}$, where $|z|^{1/3}$ denotes the unique positive cube root of the positive real number z (positive since the curves C_R and C'_ϵ do not pass through the origin). Thus, parameterising C_R as Re^{it}, $t \in [\theta_0, 2\pi - \theta_0]$, we have (note that the equation $|z^{1/3}| = |z|^{1/3}$ does not depend on which branch of the cube root function is used to calculate $z^{1/3}$, and thus we do not need to worry here and in the next integral whether the point as parameterised has an angle lying within the interval $(0, 2\pi)$)

$$\begin{aligned}\left|\int_{C_R} \frac{z^{1/3}}{(1+z^2)^2}\,dz\right| &\le \int_{\theta_0}^{2\pi-\theta_0} \frac{R^{1/3}}{|1+R^2e^{2it}|^2} R\,dt\\ &\le \int_{\theta_0}^{2\pi-\theta_0} \frac{R^{4/3}}{(R^2-1)^2}\,dt = (2\pi - 2\theta_0)\frac{R^{4/3}}{(R^2-1)^2} \to 0\end{aligned}$$

as $R \to \infty$, since the degree of the denominator is greater than that of the numerator. Similarly, parameterising C'_ϵ as ϵe^{-it}, $t \in [\pi/2, 3\pi/2]$, we have

$$\begin{aligned}\left|\int_{C'_\epsilon} \frac{z^{1/3}}{(1+z^2)^2}\,dz\right| &\le \int_{\pi/2}^{3\pi/2} \frac{\epsilon^{1/3}}{|1+\epsilon^2 e^{-2it}|^2}\epsilon\,dt\\ &\le \int_{\pi/2}^{3\pi/2} \frac{\epsilon^{4/3}}{(1+\epsilon^2)^2}\,dt = \pi\frac{\epsilon^{4/3}}{(1+\epsilon^2)^2} \to 0\end{aligned}$$

* It may seem like a very bad idea to take the branch cut along the positive real axis as, after all, this is exactly the line along which we wish to integrate! As we shall see shortly, though, this is exactly the right place to put the branch cut in this case.

as $\epsilon \to 0^+$, since the denominator goes to 1 and the numerator to 0. This completes the demonstration of (2).

We thus only need to consider how the integrals over L_R and L'_R relate to (1). We parameterise L_R by $t + i\epsilon$ and L'_R by $(R-t) - i\epsilon$, where in both cases $t \in [0, R]$. Since $\epsilon > 0$, we may express these numbers in polar notation as (here arctan gives the angle in $(-\pi/2, \pi/2)$ which has the given tangent)

$$t + i\epsilon = \sqrt{t^2+\epsilon^2}e^{i \arctan \epsilon/t}, \tag{3}$$

$$\begin{aligned}(R-t) - i\epsilon &= \sqrt{(R-t)^2+\epsilon^2}e^{i\arctan(-\epsilon/(R-t))} = \sqrt{(R-t)^2+\epsilon^2}e^{-i\arctan \epsilon/(R-t)} \\ &= \sqrt{(R-t)^2+\epsilon^2}e^{i(2\pi - \arctan \epsilon/(R-t))},\end{aligned} \tag{4}$$

where the 2π in (4) was added to make the angle lie in the interval $(0, 2\pi)$ corresponding to our chosen branch of $z^{1/3}$. Given this, then, the integrals over L_R and L'_R become

$$\int_{L_R} \frac{z^{1/3}}{(1+z^2)^2}\,dz = \int_0^R \frac{(t^2+\epsilon^2)^{1/6}e^{i\frac{1}{3}\arctan\frac{\epsilon}{t}}}{(1+(t+i\epsilon)^2)^2}\,dt,$$

$$\int_{L'_R} \frac{z^{1/3}}{(1+z^2)^2}\,dz = -\int_0^R \frac{[(R-t)^2+\epsilon^2]^{1/6}e^{i\frac{1}{3}(2\pi - \arctan\frac{\epsilon}{R-t})}}{(1+(R-t-i\epsilon)^2)^2}\,dt.$$

If we take the limit of these expressions as $\epsilon \to 0^+$ and interchange it with the integrals, we obtain, since $\arctan 0 = 0$,

$$\lim_{\epsilon\to 0^+}\int_{L_R} \frac{z^{1/3}}{(1+z^2)^2}\,dz = \int_0^R \frac{t^{1/3}}{(1+t^2)^2}\,dt,$$

$$\lim_{\epsilon\to 0^+}\int_{L'_R} \frac{z^{1/3}}{(1+z^2)^2}\,dz = -\int_0^R \frac{(R-t)^{1/3}e^{\frac{2\pi i}{3}}}{(1+(R-t)^2)^2}\,dt = -e^{\frac{2\pi i}{3}}\int_0^R \frac{t^{1/3}}{(1+t^2)^2}\,dt,$$

and we see finally that

$$\lim_{R\to\infty}\lim_{\epsilon\to 0^+}\int_\gamma \frac{z^{1/3}}{(1+z^2)^2}\,dz = \left(1 - e^{\frac{2\pi i}{3}}\right)\int_0^\infty \frac{x^{1/3}}{(1+x^2)^2}\,dx,$$

so that the integral (1) is equal to

$$\int_0^\infty \frac{x^{1/3}}{(1+x^2)^2}\,dx = \frac{2\pi i}{1 - e^{\frac{2\pi i}{3}}}\left[\mathrm{Res}_i \frac{z^{1/3}}{(1+z^2)^2} + \mathrm{Res}_{-i}\frac{z^{1/3}}{(1+z^2)^2}\right]. \tag{5}$$

We are thus left with the task of computing the residues. Note that both $\pm i$ are poles of order 2 of the function $z^{1/3}/(1+z^2)^2$; thus letting $\lambda = \pm i$, we may write

$$\mathrm{Res}_\lambda \frac{z^{1/3}}{(1+z^2)^2} = \lim_{z\to\lambda}\frac{d}{dz}(z-\lambda)^2\frac{z^{1/3}}{(1+z^2)^2}.$$

Since $1 + z^2 = (z-i)(z+i)$, this allows us to write

$$\begin{aligned}\mathrm{Res}_i \frac{z^{1/3}}{(1+z^2)^2} &= \lim_{z\to i}\frac{d}{dz}\frac{z^{1/3}}{(z+i)^2} = \lim_{z\to i}\frac{\frac{1}{3z^{2/3}}(z+i)^2 - 2(z+i)z^{1/3}}{(z+i)^4} \\ &= \frac{\frac{-4}{3e^{i\pi/3}} - 4ie^{i\pi/6}}{16} = -\frac{1}{4}\left[\frac{1}{3}e^{-i\pi/3} + ie^{i\pi/6}\right] \\ &= -\frac{1}{4}\left[\frac{1}{3}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + i\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\right] = -\frac{1}{4}\left[\left(\frac{1}{3} - 1\right)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right] \\ &= \frac{1}{6}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right),\end{aligned}$$

$$\begin{aligned}\mathrm{Res}_{-i}\frac{z^{1/3}}{(1+z^2)^2} &= \lim_{z\to -i}\frac{d}{dz}\frac{z^{1/3}}{(z-i)^2} = \frac{1}{16}\left[\frac{1}{3e^{i\pi}}(-4) + 4ie^{i\pi/2}\right] \\ &= \frac{1}{4}\left(\frac{1}{3} - 1\right) = -\frac{1}{6},\end{aligned}$$

so

$$\mathrm{Res}_i \frac{z^{1/3}}{(1+z^2)^2} + \mathrm{Res}_{-i} \frac{z^{1/3}}{(1+z^2)^2} = \frac{1}{6}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{1}{6}e^{i\pi/3}.$$

At this point we may start to feel like something has gone wrong since our answer at the end of the day needs to be a real number, but it is not clear how we shall get a real number out of what we have so far. Everything does work out in the end, though: by (5) we have

$$\begin{aligned}\int_0^\infty \frac{x^{1/3}}{(1+x^2)^2}\,dx &= \frac{2\pi i}{1-e^{\frac{2\pi i}{3}}}\left[-\frac{1}{6}e^{\frac{i\pi}{3}}\right] \\ &= -\frac{\pi}{6}\frac{2i}{e^{-i\pi/3}-e^{i\pi/3}} = \frac{\pi}{6}\sin\frac{\pi}{3} = \frac{\pi\sqrt{3}}{12},\end{aligned}$$

which is thus our final answer.

[Deep breath!]

Note how taking the branch cut along the line of integration helped us: since branch points are singularities which are not poles, whatever closed contour we draw in the complex plane must exclude the branch point and the branch cut; and as we bring the edges of the keyhole contour (the lines L_R and L_R') together, the discontinuity of $z^{1/3}$ across the branch cut will allow us to combine the two integrals without cancellation, to get a multiple of the integral along the branch cut. Thus it is precisely a multiple of the integral along the branch cut which is equal to the sum of the residues of the integrand at its poles within the contour.

Let us consider one more way of closing the contour.

Example. Evaluate

$$\int_0^\infty \frac{1}{1+x^3}\,dx. \tag{6}$$

This integral is much simpler than the previous one! It also demonstrates a slightly different application of the type of 'wedge contour' we saw when evaluating the Fresnel integrals previously. We start out with the line L_R which is just the interval $[0,R]$ in this case (since we have no branch cut!). We then consider closing the contour using a circular arc C_R of angle α followed by a line L_R' back to the origin (necessarily therefore making an angle α with the positive real axis). Now we wish the integral over L_R' to be related somehow to the original integral (6). Parameterising it as $(R-t)e^{i\alpha}$, we have

$$\int_{L_R'} \frac{1}{1+z^3}\,dz = -\int_0^R \frac{1}{1+(R-t)^3e^{3i\alpha}}e^{i\alpha}\,dt = -\int_0^R \frac{1}{1+t^3e^{3i\alpha}}e^{i\alpha}\,dt.$$

For this to be a multiple of (6), we must have $3\alpha = 2n\pi$ for some integer n. Since we wish also to include as few residues as possible in the closed contour, we want α to be as small as possible, and therefore take $\alpha = 2\pi/3$. Thus we will close using the contour shown in the figure. Now since $R/|1+z^3| \to 0$ as $R\to\infty$, when $z \in C_R$, we must have

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{1+z^3}\,dz = 0.$$

Now the only poles of the integrand in (6) are at the complex cube roots of -1, which are $e^{i\pi/3}$, $e^{i\pi}$, and $e^{5\pi/3}$; only the first of these lies within the contour $L_R + C_R + L_R'$, and thus we may write

$$(1-e^{2\pi i/3})\int_0^\infty \frac{1}{1+x^3}\,dx = 2\pi i \mathrm{Res}_{\pi i/3}\frac{1}{1+z^3}. \tag{7}$$

Now this pole is simple, so the residue can be calculated as follows:

$$\mathrm{Res}_{\pi i/3}\frac{1}{1+z^3} = \lim_{z\to\pi i/3}\frac{z-\pi i/3}{1+z^3} = \lim_{z\to\pi i/3}\frac{z-\pi i/3}{(1+z^3)-(1+[e^{i\pi/3}]^3)} = \left(\frac{d}{dz}(1+z^3))\bigg|_{z=e^{\pi i/3}}\right)^{-1};$$

compare our result on p. 6 of the lecture notes for July 14 – 16. This evaluates to

$$\frac{1}{3}e^{-2\pi i/3},$$

and so by (7) we have

$$\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi i e^{-2\pi i/3}}{3(1-e^{2\pi i/3})} = \frac{\pi}{3}\frac{2i}{e^{-\pi i/3}-e^{\pi i/3}}e^{-\pi i} = \frac{\pi}{3}\sin\frac{\pi}{3} = \frac{\pi\sqrt{3}}{6}.$$

That this integral is exactly twice that in the previous example, is a complete coincidence. (As far as I know! – I picked both integrals out of a hat, pretty much.)

The idea in this method can be combined with that in the previous example – i.e., we can make a contour consisting of two lines at angles to each other, such that the integrals over these lines can be written in terms of each other, as well as a large circle and a small circle around the origin if that happens to be a branch point. This idea is useful on question 2 of the August 3 – 7 homework.

There is one more method for turning definite integrals into contour integrals which we shall have use for; see Goursat, §45. We, again, show this by way of an example.

Example. Evaluate the integral

$$\int_0^{2\pi} \frac{1}{2-\sin x}\,dx. \tag{8}$$

We begin by noting that the integrand is continuous on the real line, so that the integral is defined. Note that this integral is fundamentally different from the other integrals we have studied so far, since it is taken over a finite interval instead of an infinite one. Thus it does not seem that the method of considering it as a contour integral and then closing the contour, as we have done previously, will be of use here. We shall instead do something completely different: rewrite (8) as a contour integral by, effectively, *deparameterising* it: in other words, recognising integral (8) as the parameterised form of an integral over a closed contour in the complex plane.

To do this, note first of all that

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i};$$

now as x ranges from 0 to 2π, e^{ix} and e^{-ix} both trace out the unit circle. Thus it would seem that integral (8) may be the parameterisation of an integral over the unit circle. Now the only other part of the integrand which depends on x is dx; if we let $z = e^{ix}$, then $dz = ie^{ix}dx = izdx$, so $dx = dz/(iz)$ (note that since z is on the unit circle, $z \neq 0$ so $1/z$ is defined), and we have finally that (letting C denote the unit circle)

$$\int_0^{2\pi} \frac{1}{2-\sin x}\,dx = \int_C \frac{1}{2-\frac{z-z^{-1}}{2i}}\frac{dz}{iz},$$

since if $z = e^{ix}$ then $e^{-ix} = 1/z$. Now we may simplify this integral as follows:

$$\int_C \frac{1}{2-\frac{z-z^{-1}}{2i}}\frac{dz}{iz} = \int_C \frac{2}{4iz-z^2+1}\,dz = -2\int_C \frac{1}{z^2-4iz-1}\,dz. \tag{9}$$

This is exactly the kind of integral which can be evaluated using the residue theorem! We just need to find the poles of the integrand. Now $z^2 - 4iz - 1 = 0$ can be solved using the quadratic formula:

$$z = 2i + (-4+1)^{1/2} = 2i + i\sqrt{3}, 2i - i\sqrt{3} = i(2 \pm \sqrt{3}).$$

Now $\sqrt{3} \in (1, 2)$, so that the root $i(2+\sqrt{3})$ lies outside the unit circle while the other root, $i(2-\sqrt{3})$, lies within it. Thus the integral (9) can be evaluated by computing the residue of the integrand at this root, which is

$$\begin{aligned}\mathrm{Res}_{i(2-\sqrt{3})}\frac{1}{z^2-4iz-1} &= \lim_{z\to i(2-\sqrt{3})}\frac{z-i(2-\sqrt{3})}{z^2-4iz-1} = \lim_{z\to i(2-\sqrt{3})}\frac{1}{z-i(2+\sqrt{3})}\\ &= \frac{1}{-2i\sqrt{3}} = \frac{i}{2\sqrt{3}},\end{aligned}$$

so that finally

$$\int_0^{2\pi} \frac{1}{2-\sin x}\,dx = -4\pi i \cdot \frac{i}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{3}.$$

Again, that this integral is *also!* an integral multiple of the previous too, is still a complete coincidence!

The method in this last example can be adapted to many other integrals with integrands which are rational functions of $\sin x$ and $\cos x$, defined everywhere on the interval of integration. See Goursat, §45, for a general discussion.

36. Liouville's Theorem. We prove the following result:

LIOUVILLE'S THEOREM. Let f be a function which is analytic and bounded on the entire complex plane. Then f must be constant.

This means that there must be a constant M such that $|f(z)| \leq M$ for all $z \in \mathbf{C}$. The proof is an easy application of the Cauchy integral formula. Let $z \in \mathbf{C}$, let $R > 0$ be any positive real number, and let C_R denote the circle of radius R centred at z. Then we must have

$$f'(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z')}{(z'-z)^2}\,dz',$$

so

$$|f'(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(z')|}{R^2} R\,dt \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{M}{R}\,dt = \frac{M}{R}.$$

But now $f'(z)$ cannot depend on R; since the right-hand side of the above expression goes to zero as $R \to \infty$, we must have $|f'(z)| = 0$, i.e., $f'(z) = 0$, for all $z \in \mathbf{C}$. This means that f must itself be constant, as claimed.

[I can't remember if we have ever proved that f' identically zero means that f must be constant when f is an analytic function. At any rate it is not hard to prove. Suppose that f is analytic on the interior of some simple closed curve, and pick any point z_0 inside that curve; then we can write, by the fundamental theorem of calculus,

$$f(z) = \int_{z_0}^{z} f'(z')\,dz' + f(z_0) = f(z_0),$$

since $f'(z') = 0$. This means that f must be constant.]

This can be used to prove that every (nonconstant) polynomial with complex coefficients has at least one complex root. To see this, let

$$P(z) = a_n z^n + \cdots + a_0,$$

where a_n, $\cdots$, $a_0 \in \mathbf{C}$ and we assume that $a_n \neq 0$. Suppose that P has no complex roots; we shall show that this implies that $n = 0$, so that P is constant. Since P has no complex roots, its reciprocal $1/P$ must be analytic everywhere on the complex plane. Now note that

$$\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n} \tag{10}$$

must go to zero as $|z|$ goes to infinity; thus there is an $R > 0$ such that the modulus of (10) is less than $\frac{1}{2}|a_n|$ when $|z| \geq R$. For such z, then, we have

$$\begin{aligned}|P(z)| &= |z|^n \left|a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right| \geq R^n \left[|a_n| - \left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right|\right] \\ &\geq R^n \left[|a_n| - \frac{1}{2}|a_n|\right] = \frac{1}{2}R^n |a_n|,\end{aligned}$$

so since $a_n \neq 0$ we must have

$$\left|\frac{1}{P(z)}\right| \leq \frac{2}{R^n |a_n|}$$

for $|z| \geq R$. Thus $1/P$ is bounded on the exterior of the disk of radius R centred at the origin. But since $|1/P|$ is a continuous function, and this disk is closed and bounded, $|1/P|$ must be bounded inside the disk as well; thus it must be bounded everywhere, and by Liouville's Theorem it must therefore be constant. Since it cannot be equal to zero, it must equal a nonzero constant, and hence $P = 1/(1/P)$ must be constant as well, as claimed.

37. The argument principle and Rouché's Theorem. We prove the following result:

THE ARGUMENT PRINCIPLE. Let C be a simple closed curve, and let f be a function analytic within and on C except possibly for poles within C, and which is nonzero on C. If Z denotes the number of zeroes and P the number of poles of f within C, counted with multiplicity, then

$$\int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i(Z - P).$$

To see this, we note the following result. If z_0 is a zero of f within C, say of multiplicity m, then there is a nonzero analytic function ϕ on a disk around z_0 such that near z_0

$$f(z) = (z - z_0)^m \phi(z);$$

while if z_0 is a pole of f within C, say of order n, then there is a nonzero analytic function ψ on a disk around z_0 such that near z_0

$$f(z) = (z - z_0)^{-n} \psi(z).$$

In the first case,

$$\frac{f'(z)}{f(z)} = \frac{m(z - z_0)^{m-1}\phi(z) + (z - z_0)^m \phi'(z)}{(z - z_0)^m \phi(z)} = \frac{m}{z - z_0} + \frac{\phi'(z)}{\phi(z)},$$

and we note that ϕ'/ϕ is analytic since ϕ is nonzero. Similarly, in the second case

$$\frac{f'(z)}{f(z)} = \frac{-n(z - z_0)^{-n-1}\psi(z) + (z - z_0)^{-n}\psi'(z)}{(z - z_0)^{-n}\psi(z)} = -\frac{n}{z - z_0} + \frac{\psi'(z)}{\psi(z)},$$

and again ψ'/ψ is analytic since ψ is nonzero. Now if z_0 is a point in C which is neither a pole nor a zero of f, then clearly f'/f is itself analytic near z_0. Thus the only poles of f'/f within C are the zeroes and the poles of f, and these are simple poles with residues equal to the multiplicitly of the zero or the negative of the order of the pole, respectively.

Now let $\{z_i\}$ and $\{p_j\}$ denote the zeroes and poles, respectively, of f within C. Then by the foregoing, and the residue theorem,

$$\begin{aligned}\int_C \frac{f'(z)}{f(z)}\,dz &= 2\pi i\left[\sum_{z_i} \operatorname{Res}_{z_i} \frac{f'(z)}{f(z)} + \sum_{p_j} \operatorname{Res}_{p_j} \frac{f'(z)}{f(z)}\right] \\ &= 2\pi i\left[\sum m_i - \sum n_j\right] = 2\pi i(Z - P),\end{aligned}$$

where m_i denotes the multiplicity of the zero z_i and n_j denotes the order of the pole p_j, and the last equation follows by the definition of Z and P.

The following result can be derived from this after a further study of the geometrical meaning of Rouché's Theorem. We shall give more details on all of this later.

ROUCHÉ'S THEOREM. Suppose that f and g are analytic within and on a simple closed curve C, and that $|f(z)| > |g(z)|$ everywhere on C. Then $f + g$ and f have the same number of zeroes inside C.

We can proceed in two different ways: by contradiction, and directly. In the lecture we gave a proof by contradiction; here we show how to proceed directly (though the direct method requires the use of the Bolzano-Weierstrass Theorem enunciated in the previous set of lecture notes). Since the polynomial P is not constant, neither is $1/P$;

Summary:

- We review the method of using analytic maps to perform a 'change of variables' in boundary-value problems involving Laplace's equation.
- We give an example, and then study the properties of a few special maps.

(Cf. Goursat, §§22, 24.)

38. Transformations of solutions to Laplace's equation. Last week we discussed the Poisson kernel and its relation to the Cauchy integral theorem: we saw that, just as the Cauchy integral formula gives the value of an analytic function everywhere inside a simple closed curve as an integral involving only the values of the function on the *boundary* of the region (i.e., the simple closed curve), so too the Poisson kernel allows us to write the values of a harmonic function inside a simple closed curve as an integral involving only the values of the function on the boundary (i.e., again, the curve).[1] Today we shall see another method of solving Laplace's equation given information about the function on the boundary, using the fact that analytic functions essentially take harmonic functions to harmonic functions and thus allow us to perform a 'change of variables' of sorts in Laplace's equation to replace a (potentially) hard problem with a (hopefully) easier one.

Before beginning this, we note one small point. When we speak of harmonic functions, we always mean real-valued functions of two real variables. On the other hand, when we speak of analytic functions on the plane we mean complex-valued functions of a complex variable. To avoid tiresome and unimportant notational issues, we shall agree that if u is any real-valued function of two real variables, and $z \in \mathbf{C}$ is any complex number, then the notation $u(z)$ (which is technically undefined) shall mean $u(\operatorname{Re} z, \operatorname{Im} z)$; in other words, we write as a shorthand

$$u(x + iy) = u(x, y).$$

With this out of the way, we have the following result, which we saw some version of back towards the start of the course: suppose that D, $E \subset \mathbf{C}$ are two regions (e.g., interiors of closed curves), that $u : D \to \mathbf{R}$ is a harmonic function on D, and that $f : E \to D$ is an analytic map with $f(E) \cap \partial D = \emptyset$.[2] Then the

[1] It is worth noting however that there is a very basic difference between these two formulas which we have so far glossed over: we have noted (though not proved) that the function u defined using the Poisson kernel approaches the function h in the limit as its argument (x, y) goes radially to a point on C. The same is *not* in general true for the Cauchy integral formula, for a very basic reason. As we noted earlier, the method by which we proved the Cauchy integral formula for derivatives allows us to prove also that any function defined as an integral of the form

$$F(z) = \int_C \frac{h(z')}{z' - z} \, dz', \tag{1}$$

where C is some simply closed curve, z is any point inside C, and h is a continuous function on C, must also have a derivative everywhere inside C, and hence must be analytic. But *this analytic function need not agree with h on the boundary curve C*, for the following reason: suppose that h were real-valued; then if $F(z) = h(z)$ for $z \in C$, the analytic function F would be real-valued on C, and hence its imaginary part would be zero everywhere on C. But its imaginary part must be harmonic, and any harmonic function which vanishes on a simple closed curve must vanish everywhere inside that curve; hence F must be real-valued inside C as well as on C, and must therefore be constant. But if h is not constant then this contradicts $F(z) = h(z)$ on C. Another way of looking at this is that the function F includes two nonzero harmonic functions, so somehow the integral (1) is actually giving us *two* harmonic functions, which clearly contain more information on the boundary than is included in the function h alone. (I should point out that in general there is no reason to believe that even the real part of F agrees with h on the boundary curve C; and here it is worth recalling that when we derived the Poisson kernel we had to add in an extra term by hand involving the point z^*.) This is a bit of a side comment, but worth keeping in mind to avoid making mistakes.

[2] In the applications we shall be mostly interested in cases where the boundary curves of D and E are simple (i.e., where D and E are simply connected), where $f(\partial E) = \partial D$, and where f^{-1} is also analytic. But nothing in this present result relies on any of these conditions, so we state it in greater generality.

function

$$v = u \circ f : E \to \mathbf{R}$$

is also harmonic on E.

To see this, let $a \in E$; then $f(a) \in D$, so that u must be harmonic at $f(a)$. Now we can find an $\epsilon > 0$ so that the disk of radius ϵ centred at $f(a)$ is still contained in D [we do not care whether it is contained in $f(E)$; that is actually totally irrelevant]; let us denote this disk by U. Thus u is actually harmonic on U, and there must then be a conjugate harmonic function on U, call it $\tilde{u}$, so that $u + i\tilde{u}$ is analytic on U. (See Goursat, §3, p. 10; or §9 in these lecture notes.) Let us denote this function by g. Then $g \circ f$ must be analytic at a, by the chain rule; but since the real and imaginary parts of all analytic functions are harmonic, its real part $u \circ f = v$ must be harmonic on E, as desired. (The restricting to a disk is only to make sure that the function $\tilde{u}$ is well-defined – as we have seen before (e.g., in the second quiz), on a non-simply connected region a conjugate harmonic function may become multiple-valued.)

For the applications, we need to restrict the regions E and D and the function f, and for this we need a bit more notation. We shall require E and D to be simply connected; equivalently, we require their boundary curves ∂D and ∂E to be simple closed curves. We shall also require E and D to represent only the interior of their boundary curves, i.e., that $E \cap \partial E = D \cap \partial D = \emptyset$; we set $\overline{E} = E \cup \partial E$, $\overline{D} = D \cup \partial D$.[3] We assume that $f^{-1} : D \to E$ (exists and) is analytic, and that f and f^{-1} extend to continuous functions mapping $\overline{E}$ to $\overline{D}$ and $\overline{D}$ to $\overline{E}$, respectively; at the risk of being extremely confusing, we shall denote these extensions by f and f^{-1} as well.

To put all of this more simply, we assume that f is invertible with analytic inverse, and that f and its inverse extend to continuous functions on the boundary of E and D, respectively.

Now suppose that u satisfies

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = h, \tag{2}$$

where h is some piecewise-continuous function on ∂D. We call any problem like (2) a *boundary-value problem,* since we have conditions on u on the boundary of the region. Now let E and $f : E \to D$ satisfy the conditions just described, and set $v = u \circ f$. Then v is harmonic on E, while if $a \in \partial E$ we must have $f(a) \in \partial D$, so that

$$v(a) = (u \circ f)(a) = u[f(a)] = h[f(a)]$$

by the boundary condition in (2). Thus v satisfies the boundary-value problem

$$\Delta v = 0 \text{ on } E, \qquad v|_{\partial E} = h \circ f. \tag{3}$$

Now it may happen that we can find E and f such that problem (3) is simpler than (2); in fact, in the examples we shall see the answer to (3) can be guessed at quite easily. Suppose thus that we can find some function v satisfying (3). Then running the above logic backwards, we see that the function $\tilde{u} = v \circ f^{-1}$ must satisfy the boundary-value problem

$$\Delta \tilde{u} = 0 \text{ on } D, \qquad \tilde{u}|_{\partial D} = h;$$

in other words, $\tilde{u}$ is a solution to our original boundary value problem (2).

Let us see an example.

Example. Solve the following boundary problem on the region D given in polar coordinates as $D = \{(r, \theta) \,|\, r \in (0, 1),\, \theta \in (-\pi/4, \pi/4)\}$:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = \begin{cases} \sin 2\theta, & r = 1 \\ r^2, & \theta = \pi/4 \\ -r^2, & \theta = -\pi/4 \end{cases} \tag{4}$$

The factor of 2 in the sine, the square in $\pm r^2$, and the shape of the region suggest that perhaps f ought to have something to do with a square. Now $z \mapsto z^2$ increases the angle by a factor of 2; thus if we were to

[3] For those who know something of topology, we note that $\overline{E}$ and $\overline{D}$, as the notation suggests, are just the closure of E and D in the usual topology of $\mathbf{R}^2$ in this case.

take $f(z) = z^2$, the region E would be even narrower than the region D, and the problem would probably not be any simpler. If we take $f(z) = z^{1/2}$, though (assuming we can take an appropriate branch!), then we see that the region E would be a half-disk. Thus let us let $f(z) = z^{1/2}$, where we take a branch cut along the negative real axis and require $\theta \in (-\pi, \pi)$. Then if we set

$$E = \{(r, \theta) \mid r \in (0,1),\, \theta \in (-\pi/2, \pi/2)\},$$

we see that $f : E \to D$ is one-to-one and onto, and its inverse is given by $z \mapsto z^2$, which is analytic on D. Further, we may extend f to a continuous function on $\overline{E}$ by setting $f(0) = 0$ (this will not extend to an analytic function, of course, but that does not matter). Now we must determine how the boundary condition changes; in other words, if we define $h : \partial D \to \mathbf{R}$ by

$$h = \begin{cases} \sin 2\theta, & r = 1 \\ r^2, & \theta = \pi/4 \\ -r^2, & \theta = -\pi/4 \end{cases},$$

then we need to determine $h \circ f : \partial E \to \mathbf{R}$. Now ∂E also has three pieces, namely $r = 1$, $\theta \in [-\pi/2, \pi/2]$, $\theta = \pi/2$, $r \in [0,1]$, and $\theta = -\pi/2$, $r \in [0,1]$. Now if $z = re^{i\theta}$, where $\theta \in (-\pi, \pi)$, then $f(z) = r^{1/2} e^{\frac{1}{2} i\theta}$; thus these three boundary pieces are mapped to, respectively,

$$r = 1,\, \theta \in [-\pi/4, \pi/4]; \quad \theta = \pi/4,\, r \in [0,1]; \quad \theta = -\pi/4,\, r \in [0,1].$$

Now if $r = 1$ and $\theta \in [-\pi/2, \pi/2]$, then we have $r^{1/2} = 1$ and

$$(h \circ f)(re^{i\theta}) = h(r^{1/2} e^{\frac{1}{2} i\theta}) = h(e^{\frac{1}{2} i\theta}) = \sin 2 \cdot \left(\frac{1}{2}\theta\right) = \sin\theta;$$

if $\theta = \pi/2$ and $r \in [0,1]$, then we have $\frac{1}{2}\theta = \pi/4$ and

$$(h \circ f)(re^{i\theta}) = h(r^{1/2} e^{\frac{1}{2} i\theta}) = \left(r^{1/2}\right)^2 = r;$$

finally, if $\theta = -\pi/2$ and $r \in [0,1]$, then we have $\frac{1}{2}\theta = -\pi/4$ and

$$(h \circ f)(re^{i\theta}) = h(r^{1/2} e^{\frac{1}{2} i\theta}) = -\left(r^{1/2}\right)^2 = -r.$$

Pulling this together, we have

$$h \circ f = \begin{cases} \sin\theta, & r = 1 \\ r, & \theta = \pi/2 \\ -r, & \theta = -\pi/2 \end{cases},$$

so that we wish to solve the problem

$$\Delta v = 0 \text{ on } E, \qquad v|_{\partial E} = \begin{cases} \sin\theta, & r = 1 \\ r, & \theta = \pi/2 \\ -r, & \theta = -\pi/2 \end{cases} \tag{5}$$

Now we note that if a, b, and c are real numbers, then the function

$$g(x, y) = a + bx + cy \tag{6}$$

will be harmonic at every point $(x, y) \in \mathbf{R}^2$, since all of its second order derivatives vanish. Let us see whether we can find a solution to (5) which is of this form. The idea is that, since all of these functions satisfy Laplace's equation, we only need to fit the boundary conditions. Now in polar coordinates, the function g may be written

$$g = a + br\cos\theta + cr\sin\theta.$$

If $r = 1$, this gives $a + b\cos\theta + c\sin\theta$, which will fit the boundary condition in (5) if $a = b = 0$, $c = 1$; if $\theta = \pm\pi/2$, then it gives

$$a \pm cr,$$

which will also fit the boundary condition in (5) if $a = 0$ and $c = 1$. In other words, we have found that the function

$$v(x, y) = y$$

is a solution to (5).

Working backwards, then, we know from our general work above that the function $u = v \circ f^{-1}$ will solve our original problem. Now $f^{-1}(z) = z^2$, so that we have finally the solution

$$u(x, y) = v[f(x + iy)] = v(x^2 - y^2, 2xy) = 2xy$$

to the boundary value problem (4).

If we step back and look at the broad sweep of the logic used in this example, we see that to make efficacious use of this technique in practice, we must have a catalogue of two things: one, transformations and regions; two, standard solutions to Laplace's equation on the domain regions of these transformations. For us, the main class of standard solutions will be the linear ones given in (6). We now describe a few elementary transformations which are useful.

EXAMPLES of transformations and regions useful for solving Laplace's equation.

1. Powers and roots on wedges. If $n \in \mathbf{Z}$, $n > 0$, then the map $z \mapsto z^n$ takes a wedge with vertex at the origin with angle α to another wedge with vertex at the origin with angle $n\alpha$. For this transformation to be invertible, the angle α must be less than $2\pi/n$. Similarly, if $n \in \mathbf{Z}$, $n > 0$, then the map $z \mapsto z^{1/n}$, after taking an appropriate branch, will map a wedge of angle α with vertex at the origin to another wedge with vertex at the origin and angle α/n, assuming that $\alpha < 2\pi$. Wedges have three boundaries, two of which are lines and the third of which is a circular arc; on the circular arc, a map $z \mapsto z^p$ ($p = n$ or $p = 1/n$, as the case may be) will take a point $re^{i\theta}$ to a point $r^p e^{ip\theta}$, i.e., it will multiply the angle θ by p (where θ must be in the range corresponding to the chosen branch if $p = 1/n$), while on the lines the map acts by simple exponentiation, as in the example above.

2. Inversion. The map $z \mapsto z^{-1}$, as we have seen before (see §16 of these lecture notes), takes the punctured plane into itself, but 'turns it inside out' by mapping the region outside the unit circle to the region inside the unit circle.

3. The exponential function. Consider the map $z \mapsto e^z$. We have seen already (see the solutions to the first homework assignment) that this map takes lines parallel to the real axis to rays from the origin, and lines parallel to the imaginary axis to circles centred at the origin. Thus this map will take certain rectangular regions in the plane into annular wedges. (Going further, to whole annular regions or even a full disk, would involve extra considerations beyond the scope of our present discussion.)

4. The maps $z \mapsto \sin z$ and $z \mapsto \cos z$. The second of these has already been described in some detail in §16 of these lecture notes (cf. Goursat, §22); in particular, it was shown there that $z \mapsto \cos z$ takes the rectangle $\{x + iy \mid x \in (0, \pi),\ y \in (0, +\infty)\}$ into the lower half-plane $\{x + iy \mid y < 0\}$. Let us see a bit more carefully what it does on the boundary of this rectangle. This boundary consists of three lines, the half-line $x = 0$, $y \geq 0$, the line segment $y = 0$, $x \in [0, \pi]$, and the half-line $x = \pi$, $y \geq 0$. Let us consider these in turn. We have (see, e.g., §13 of these notes)

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y;$$

thus if $x = 0$ we have

$$\cos(iy) = \cosh y,$$

which takes the interval $[0, +\infty)$ into the interval $[1, +\infty)$ in a one-to-one fashion. Similarly, if $x = \pi$, then we have

$$\cos(\pi + iy) = -\cosh y,$$

which takes the interval $[0, +\infty)$ into the interval $(-\infty, -1]$. Finally, if $y = 0$, $\cos z$ will be just the ordinary real-valued function $\cos x$, which takes the interval $[0, \pi]$ into the interval $[-1, 1]$ (though in reverse order, i.e., it is decreasing on that interval). All told, then, the half-line $x = 0$, $y \geq 0$ will map to the interval $[1, +\infty)$, the segment $y = 0$, $x \in [0, \pi]$ will map to the interval $[-1, 1]$ (where the point 1 is mapped from the intersection $x = y = 0$ of these two parts of the boundary) and finally the half-line $x = \pi$, $y \geq 0$ will map to the interval $(-\infty, -1]$, with as before the point -1 is mapped from the intersection $x = \pi$, $y = 0$ of these last two parts of the boundary. This suggests that this map may be useful if we are interested in finding solutions to problems on the lower half-plane whose initial data can be broken down in some way across the three intervals $(-\infty, -1]$, $[-1, 1]$, and $[1, +\infty)$.

We note one other result: if $x = \pi/2$, then we have

$$\cos(x + iy) = \cos(\pi/2 + iy) = -i \sinh y,$$

from which we see that the half-line $x = \pi/2$, $y \geq 0$ is mapped to the negative imaginary axis. This means, incidentally, that the two rectangles

$$\{x + iy \,|\, x \in (0, \pi/2),\, y > 0\}, \quad \{x + iy \,|\, x \in (\pi/2, \pi),\, y > 0\}$$

map to the fourth and third quadrants, respectively (since $\cos x > 0$ for $x \in (0, \pi/2)$ and $\cos x < 0$ for $x \in (0, \pi/2)$). Thus this map can also be used for problems on a quarter-plane.

Similarly, let us consider the map $z \mapsto \sin z$. We have (ibid.)

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y.$$

In this case we take as our domain the rectangular strip $\{x + iy \,|\, x \in (-\pi/2, \pi/2),\, y > 0\}$, since $\sin x$ is invertible on $(-\pi/2, \pi/2)$ but not on any strictly larger interval. Let us consider the values of sin on the three boundary lines as we did for cos. If $x = -\pi/2$, then we have

$$\sin(-\pi/2 + iy) = -\cosh y,$$

while if $x = \pi/2$ then we have

$$\sin(\pi/2 + iy) = \cosh y;$$

thus these two lines map to the segments $(-\infty, -1]$ and $[1, +\infty)$, similarly to what we found for cos (except note that the order is reversed). Similarly, if $y = 0$ then $\sin z$ is just the ordinary real-valued sine function $\sin x$, which maps the interval $[-\pi/2, \pi/2]$ to the interval $[-1, 1]$. Thus, again, the boundary is mapped onto the entire real line. Since $\cos x$ and $\sinh y$ are both positive everywhere on the rectangle, we see that sin maps the rectangle into the upper half-plane; and this map is actually onto. Again, as with cos, we see that the midline $x = 0$, $y \geq 0$ maps to the positive real axis, and we see further that the rectangles

$$\{x + iy \,|\, x \in (-\pi/2, 0),\, y > 0\}, \quad \{x + iy \,|\, x \in (0, \pi/2)\, y > 0\}$$

map to the second and first quadrants, respectively (again, which quadrant is the image of which rectangle can be determined from the sign of x).

Summary:

- We give a proof of L'Hôpital's rule for analytic functions.
- We then show how another class of boundary conditions for Laplace's equation transforms under analytic maps, and give an example.
- [For the material on analytic continuation, please see the pre-class notes.]

39. L'Hôpital's rule for analytic functions. We prove the following version of L'Hôpital's rule for analytic functions. Suppose that f and g are analytic near a point a, and that both f and g have a zero of order m at a. Then

$$\lim_{z\to a}\frac{f(z)}{g(z)}=\lim_{z\to a}\frac{f^{(m)}(z)}{g^{(m)}(z)}. \tag{1}$$

(In the terminology usually used to discuss L'Hôpital's rule in elementary calculus courses, this is a limit of type 0/0.)

To see this, note that since f and g have zeroes of order m at a, there are functions $\phi(z)$ and $\gamma(z)$ which are both analytic and nonzero near a and satisfy

$$f(z)=(z-a)^m\phi(z),\qquad g(z)=(z-a)^m\gamma(z).$$

Moreover, we claim that $\phi(a)=f^{(m)}(a)/m!$ and $\gamma(a)=g^{(m)}(a)/m!$. The proof of these two equations is the same so we show only the first one. There are two distinct ways of doing this. The one we used in lecture involved Taylor series and is as follows. Suppose that the Taylor series for ϕ at a is

$$\sum_{k=0}^{\infty}a_k(z-a)^k,$$

where we know that $a_0=\phi(a)\neq 0$. Then the Taylor series for f at a is

$$\sum_{k=0}^{\infty}a_k(z-a)^{k+m}.$$

But this series must equal

$$\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(z-a)^n;$$

equating coefficients of like powers, we see that

$$a_0=\frac{1}{m!}f^{(m)}(a),$$

as claimed.

The second method is much quicker and involves the Cauchy integral formula: if C is a sufficiently small circle around a, then we have

$$\begin{aligned}\phi(a)&=\frac{1}{2\pi i}\int_C\frac{\phi(z')}{z'-a}\,dz'\\&=\frac{1}{2\pi i}\int_C\frac{f(z')/(z'-a)^m}{z'-a}\,dz'=\frac{1}{2\pi i}\int_C\frac{f(z')}{(z'-a)^{m+1}}\,dz'\\&=\frac{1}{m!}f^{(m)}(a),\end{aligned}$$

by the Cauchy integral formula for derivatives.

Given this, the proof of (1) is easy:

$$\lim_{z\to a}\frac{f(z)}{g(z)}=\lim_{z\to a}\frac{(z-a)^m\phi(z)}{(z-a)^m\gamma(z)}=\lim_{z\to a}\frac{\phi(z)}{\gamma(z)}=\frac{\phi(a)}{\gamma(a)}=\frac{f^{(m)}(a)}{g^{(m)}(a)},$$

since the factors of $m!$ will cancel.

We can also use the inner workings of the above proof to evaluate limits, which is often much easier than applying the result itself, as the following example shows.

EXAMPLE. Let $n \in \mathbf{Z}$ be positive. Find

$$\lim_{x\to 0} \frac{(\sin x)^{2n}}{(1-\cos x)^n}.$$

Let us consider the corresponding complex limit, i.e., replace the real number x in the limit above by a complex number z, since if the resulting limit exists then certainly the original limit does as well. Now note that $\sin z$ has a zero of order 1 at 0, so that we may write

$$\sin z = z\phi(z)$$

for some function ϕ which will be analytic and nonzero at $z = 0$ (and hence everywhere in this case, though that is not important). In fact, of course, the function ϕ will simply be the function

$$\begin{cases} \sin z/z, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

which we have seen many times already. We have moreover that $\phi(0) = \frac{d}{dz}\sin z\big|_{z=0} = 1$. Further, since

$$\cos z = 1 - \frac{1}{2}z^2 + \cdots, \tag{2}$$

we see that $1 - \cos z$ has a zero of order 2 at 0, so we may write

$$1 - \cos z = z^2\gamma(z),$$

where by inspection of the series (2) we have $\gamma(0) = \frac{1}{2}$. (This could also be obtained by differentiating as in the proof above, of course; in that case the factor of $1/2$ comes from the $1/m!$.) Thus we may write

$$\begin{aligned} \lim_{z\to 0} \frac{(\sin z)^{2n}}{(1-\cos z)^n} &= \lim_{z\to 0} \frac{z^{2n}[\phi(z)]^{2n}}{z^{2n}[\gamma(z)]^n} \\ &= \lim_{z\to 0} \frac{[\phi(z)]^{2n}}{[\gamma(z)]^n} = \frac{[\phi(0)]^{2n}}{[\gamma(0)]^n} = 2^n. \end{aligned}$$

To apply L'Hôpital's rule to this directly would have required us to differentiate n times, which would be extremely messy. This demonstrates the utility of power series manipulations and other concepts (such as the order of a zero) which we have studied in this course in elucidating the behaviour of analytic functions.

40. Transformation of Neumann boundary conditions by analytic maps. So far we have considered only problems involving so-called *Dirichlet* boundary conditions, i.e.,

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = h. \tag{3}$$

This means, of course, that we seek a function u which satisfies Laplace's equation inside the region D and is equal to the function h on the boundary of D; note that the Laplacian of u need not be defined on the boundary of D. Another type of boundary condition we could give is that the normal deriviative of u on the boundary be equal to some given function. In other words, let $\mathbf{n}$ denote the unit outward normal to the boundary curve ∂D (this is defined in an analogous way to how one defines the outward unit normal to a region in three-dimensional space when one discusses the divergence theorem in multivariable calculus); then, writing

$$\frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u$$

(i.e., we *define* $\partial u/\partial n$ to be the quantity on the right-hand side), we may consider the problem

$$\Delta u = 0 \text{ on } D, \qquad \frac{\partial u}{\partial n}\bigg|_{\partial D} = h, \tag{4}$$

where again h is some function on the boundary ∂D. Note first of all that this problem will not have a unique solution on D, since if u is any solution then clearly so also is $u+C$, where C is any real number; thus in order to get a unique solution we need to specify some other piece of information about u, for example, its value at some point. We shall assume that given the value of u at any point, (4) has a unique solution. We shall also restrict attention to the case $h = 0$.

Let us see whether our method of transforming the problem with an analytic function is applicable to problem (4). Thus consider a region E and an analytic function $f : E \to D$ satisfying the conditions given in §38 of these lecture notes, and consider problem (4) with $h = 0$. We claim that if $v = u \circ f$, then v satisfies the problem

$$\Delta v = 0 \text{ on } E, \qquad \left.\frac{\partial v}{\partial N}\right|_{\partial E} = 0, \tag{5}$$

where $\partial/\partial N$ denotes the derivative of v in the direction normal to E. Let us see why this is so. That v satisfies $\Delta v = 0$ has already been shown; thus we need only consider the boundary condition. Let $a \in E$. Let $\mathbf{N}$ denote the outward unit normal to E at a; then

$$\left.\frac{\partial v}{\partial N}\right|_a = \mathbf{N} \cdot \nabla v(a) = \mathbf{N} \cdot \nabla(u \circ f)(a).$$

Now let $b = f(a)$ and consider $\nabla u(b)$. If $\nabla u(b) = 0$, then it is not hard to show that $\nabla(u \circ f)(a) = 0$ (using the chain rule); thus in this particular case $\partial v/\partial N|_a$ will be zero, as claimed. Thus now suppose that $\nabla u(b) \neq 0$. Since

$$\left.\frac{\partial u}{\partial n}\right|_b = \mathbf{n} \cdot \nabla u(b) = 0,$$

where $\mathbf{n}$ denotes the outward unit normal to D at $b = f(a)$, we see that $\nabla u(b)$ must be perpendicular to $\mathbf{n}$. But now $\nabla u(b)$ must also be perpendicular to the level curve, call it C, of u which passes through b, i.e., to the set of points (x, y) satisfying $u(x, y) = u(b)$; since we are in the plane this implies that this level curve must be tangent to $\mathbf{n}$ and hence perpendicular to the boundary curve ∂D. But since f^{-1} is assumed to be analytic and therefore conformal (its derivative will be nonzero since it is invertible), and maps ∂D to ∂E, this means that the curve C' to which f^{-1} maps C must be perpendicular to the boundary ∂E at the point a. But this curve is just the level curve of v through a, for (x, y) satisfies $v(x, y) = v(a)$ if and only if

$$u[f(x, y)] = u[f(a)] = u(b),$$

i.e., if and only if $f(x+iy) = x'+iy'$ for some point (x', y') in C, which is equivalent to $x+iy = f^{-1}(x'+iy')$, i.e., that (x, y) lie on C'. Thus the level curve of v through a must be perpendicular to the boundary ∂E, which means as before that the gradient of v at a must be perpendicular to the normal vector $\mathbf{N}$, and hence

$$\left.\frac{\partial v}{\partial N}\right|_a = \mathbf{N} \cdot \nabla v(a) = 0,$$

so that v satisfies the boundary condition in (5), as claimed.

Before going into the example, recall that the map $z \mapsto \sin z$ was described at the end of §38 in the previous set of lecture notes.

Example. Let $U = \{(x, y)|y > 0\}$ denote the upper half-plane. Solve the following problem:

$$\Delta u = 0 \text{ on } U, \qquad u|_{\partial U} = \begin{cases} 0, & x \in (-\infty, -1) \\ 1 & x \in (1, \infty) \end{cases}, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial U} = 0 \text{ if } x \in [-1, 1]; \tag{6}$$

in other words, we require u to be 0 on the real axis except on the interval $[-1, 1]$, and we require $\partial u/\partial n$ to be zero on this interval. Now geometrically it is clear that $\partial u/\partial n = -\partial u/\partial y$, so that this is equivalent to requiring $\partial u/\partial y$ to vanish on this interval.

Now of the conformal maps listed at the end of §38, the only one which maps onto the upper half-plane is $z \mapsto \sin z$, where we take z to lie in the rectangle

$$R = \{(x, y)|x \in (-\pi/2, \pi/2), \ y > 0\}.$$

Thus we consider transforming the problem (6) using this map. Let $v(z) = u[\sin(z)]$; then v must satisfy the problem

$$\nabla v = 0 \text{ on } R, \qquad v(-\pi/2, y) = 0, \qquad v(\pi/2, y) = 1, \qquad \left.\frac{\partial v}{\partial N}\right|_{(x,0)} = 0,\ x \in (-\pi/2, \pi/2), \tag{7}$$

where here $\partial/\partial N$ denotes the outward normal derivative of v; but as before this is just $-\partial/\partial y$, so that this last condition gives simply

$$\left.\frac{\partial v}{\partial y}\right|_{(x,0)} = 0,\ x \in (-\pi/2, \pi/2).$$

Thus we must now find a function v which satisfies (7). Since the only real class of solutions we have on rectangles are the linear ones, let us see whether this problem has a linear solution; in other words, let us see whether we can find numbers a, b, and c such that

$$v(x, y) = a + bx + cy$$

(which will automatically satisfy $\Delta u = 0$) satisfies the boundary conditions. The conditions on $x = \pm\pi/2$ give

$$0 = v(-\pi/2, y) = a - b\pi/2 + cy, \qquad 1 = v(\pi/2, y) = a + b\pi/2 + cy,$$

whence we see immediately that $c = 0$ (as otherwise the right-hand sides of these expressions would depend on y while the left-hand sides clearly do not) and are thus left with the system

$$a - \frac{\pi}{2}b = 0$$
$$a + \frac{\pi}{2}b = 1.$$

Adding these equations gives $a = 1/2$, while subtracting the first from the second gives $b = 1/\pi$; thus the function

$$v = \frac{1}{2} + \frac{1}{\pi}x$$

will indeed satisfy the boundary conditions on $x = \pm\pi/2$, a fact which can readily be verified by direct substitution. Note that this solution satisfies also $\partial v/\partial y = 0$ everywhere, and hence in particular on the segment $\{(x, 0) \mid x \in (-1, 1)\}$, which is the final boundary condition in (7). Thus v is the solution to (7).

This means that the solution to the original problem will be $u = v \circ \arcsin z$. Now if we write a point in the plane in complex notation, then we have $v(x + iy) = \frac{1}{2} + \frac{1}{\pi}x = \frac{1}{2} + \frac{1}{\pi}\mathrm{Re}\,(x + iy)$; thus we need to determine $\mathrm{Re} \arcsin z$, where z is in the upper half-plane and the branch of arcsin taken is that which maps into the rectangle R. We could determine this using the formula for arcsin which we found previously, but it seems easier to proceed in a different way. Let $z = a + ib$ be in the upper half-plane, and let $w = x + iy \in R$ satisfy $\sin w = z$; in other words,

$$\sin w = \sin x \cosh y + i \cos x \sinh y = a + ib,$$

and we note that $b > 0$, which implies that $x \in (-\pi/2, \pi/2)$ and $y > 0$. We wish to find $\mathrm{Re}\, w = x$. Now as we just saw, $b > 0$ implies $\cos x > 0$. Further, $\sin x = 0$ exactly when $a = 0$, and in this case we must have $x = 0$. Thus for us $\mathrm{Re}\, arcsin 0 = 0$, and we may assume that $a \neq 0$. Now let $\alpha = \sin^2 x$. Then $\cos^2 x = 1 - \alpha$, so that we have

$$\frac{a^2}{\alpha} + \frac{b^2}{\alpha - 1} = \frac{a^2}{\sin^2 x} - \frac{b^2}{\cos^2 x} = \cosh^2 y - \sinh^2 y = 1;$$

if we multiply through by $\alpha(\alpha - 1)$, this gives

$$\alpha^2 - \alpha = a^2(\alpha - 1) + \alpha b^2 = \alpha(a^2 + b^2) - a^2$$
$$\alpha^2 - (1 + a^2 + b^2)\alpha + a^2 = 0$$

whence applying the quadratic formula, we obtain

$$\alpha = \frac{1}{2}\left[1 + a^2 + b^2 \pm \sqrt{(1 + a^2 + b^2)^2 - 4a^2}\right]. \tag{8}$$

That the quantity inside the square root is always positive can be seen as follows:

$$\begin{aligned}(1 + a^2 + b^2)^2 - 4a^2 &= 1 + 2a^2 + 2b^2 + a^4 + 2a^2b^2 + b^4 - 4a^2 \\ &= b^2(b^2 + 2) + a^4 - 2a^2 + 1 + 2a^2b^2 = b^2(b^2 + 2) + 2a^2b^2 + (a^2 - 1)^2,\end{aligned}$$

which is a sum of positive (since $b \neq 0$) and nonnegative quantities and therefore positive. Now note that both roots above are nonnegative, since $\sqrt{(1 + a^2 + b^2)^2 - 4a^2} \leq 1 + a^2 + b^2$. We claim that the + root in (8), which we denote by α_+, must be greater than 1. To see this, note that since $b > 0$

$$4 - 4(1 + b^2) = -4b^2 < 0;$$

thus

$$(1+b^2+a^2)^2 - 4a^2 > (1+b^2+a^2)^2 - 4(a^2+b^2) = (1+b^2+a^2)^2 - 4(1+a^2+b^2) + 4 = [(1+b^2+a^2) - 2]^2,$$

so

$$\sqrt{(1 + b^2 + a^2)^2 - 4a^2} > \left|1 + b^2 + a^2 - 2\right|.$$

Now if $1 + b^2 + a^2 > 2$, then clearly $\alpha_+ > 1$; while if $1 + b^2 + a^2 \leq 2$, then by the above inequality we have

$$\sqrt{(1 + b^2 + a^2)^2 - 4a^2} > 2 - (1 + b^2 + a^2), \qquad (1 + b^2 + a^2) + \sqrt{(1 + b^2 + a^2)^2 - 4a^2} > 2,$$

and again $\alpha_+ > 1$. Thus in any event we must have $\alpha_+ > 1$, as desired. Since we require $\alpha = \sin^2 x$ for x real, we are only interested in values of α that lie in $[0, 1]$, and we therefore reject the value α_+ and take $\alpha = \frac{1}{2}\left[1 + a^2 + b^2 - \sqrt{(1 + a^2 + b^2)^2 - 4a^2}\right]$. We claim that $\alpha < 1$. This can be seen as follows:

$$\begin{aligned}2\alpha = 1 + a^2 + b^2 - \sqrt{(1 + a^2 + b^2)^2 - 4a^2} &= \frac{(1 + a^2 + b^2)^2 - [(1 + a^2 + b^2)^2 - 4a^2]}{1 + a^2 + b^2 + \sqrt{(1 + a^2 + b^2)^2 - 4a^2}} \\ = \frac{4a^2}{1 + a^2 + b^2 + \sqrt{b^2(b^2 + 2) + 2a^2b^2 + (a^2 - 1)^2}} &< \frac{4a^2}{1 + a^2 + b^2 + a^2 - 1} < \frac{4a^2}{2a^2} = 2,\end{aligned}$$

giving $\alpha < 1$, as desired. (Here we have used $b > 0$ to conclude that $\sqrt{b^2(b^2 + 2) + 2a^2b^2 + (a^2 - 1)^2} > \sqrt{(a^2 - 1)^2} = |a^2 - 1| \geq a^2 - 1$.) This gives, finally, then,

$$x = \text{Re} \arcsin z = \pm \arcsin \sqrt{\frac{1}{2}\left[1 + a^2 + b^2 - \sqrt{(1 + a^2 + b^2)^2 - 4a^2}\right]},$$

where here arcsin denotes the ordinary inverse sine of a number in the interval $[0, 1)$, lying in $[0, \pi/2)$. We will take the + sign for $a \geq 0$ and the − for $a \leq 0$; $a = 0$ clearly gives $x = 0$, so in this case it does not matter which sign we take. Our solution is then finally

$$u(x, y) = \frac{1}{2} + \frac{1}{\pi} \cdot \begin{cases} \arcsin \sqrt{\frac{1}{2}\left[1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2}\right]}, & x \geq 0 \\ -\arcsin \sqrt{\frac{1}{2}\left[1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2}\right]}, & x \leq 0 \end{cases}. \tag{9}$$

It is instructive to see how this solution satisfies the boundary conditions. Suppose $y = 0$; then the formula above gives[1]

$$1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2} = 1 + x^2 - \sqrt{(1 - x^2)^2} = 1 + x^2 - |1 - x^2|.$$

[1] Technically, of course, we should be considering rather the *limit* as $y \to 0^+$. The formula above will clearly be continuous at $y = 0$, however, if it is defined at $y = 0$; and thus it suffices to show that it is defined at $y = 0$, which is afforded by our following calculation.

Now if $x \in [-1, 1]$, this gives $1 + x^2 - (1 - x^2) = 2x^2$, while if $x \in (-\infty, -1) \cup (1, +\infty)$ it gives instead $1 + x^2 - (x^2 - 1) = 2$. Thus

$$\arcsin \sqrt{\frac{1}{2}\left[1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2}\right]} = \begin{cases} \arcsin |x|, & x \in [-1, 1] \\ \frac{\pi}{2}, & x \in (-\infty, -1) \cup (\infty, 1) \end{cases},$$

where the second line follows from $\arcsin 1 = \pi/2$. If we substitute this back into (9), we thus obtain (using the fact that arcsin is odd)

$$\begin{aligned} u(x, 0) &= \frac{1}{2} + \frac{1}{\pi} \begin{cases} -\frac{\pi}{2}, & x \in (-\infty, -1) \\ \arcsin x, & x \in [-1, 1] \\ \frac{\pi}{2}, & x \in (1, +\infty) \end{cases} \\ &= \begin{cases} 0, & x \in (-\infty, -1) \\ \frac{1}{2} + \frac{1}{\pi} \arcsin x, & x \in [-1, 1] \\ 1, & x \in (1, +\infty) \end{cases}, \end{aligned}$$

which clearly satisfies the boundary conditions on $(-\infty, -1) \cup (\infty, 1)$. That the remaining condition on $(-1, 1)$ holds, we leave to the intrepid reader; it follows fairly simply from the observation that

$$\frac{\partial}{\partial y}\left[1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2}\right] = 2y - 2y(1 + x^2 + y^2)[(1 + x^2 + y^2)^2 - 4x^2]^{-1/2},$$

and the observation that this quantity vanishes as $y \to 0^+$ for all $x \in (-1, 1)$. (This is clear, since the expression inside the exponent is positive for all such x even when $y = 0$, by the foregoing.)

Summary:

- We motivate the definition of the Poisson kernel.

101. Poisson kernel . In lecture we attempted to give a *derivation* of the Poisson kernel. While the main idea was correct, there were a few errors in detail and interpretation, and when those are corrected it turns out that what was given can *motivate* the definition of the Poisson kernel, but does not really serve as a proof. We go through this motivation anyway.

Let D be the disk of radius r centred at the origin, $C = \partial D$ its boundary (the unit circle centred at the origin), and consider the following problem:

$$\Delta u = 0, \qquad u|_{\partial D} = h, \tag{1}$$

where Δ is the Laplacian, $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$, and h is a function on C. Now suppose that there is a function f which is analytic on the complex plane and satisfies $\operatorname{Re} f|_{\partial D} = h$; since $\operatorname{Re} f$ must be harmonic everywhere on the plane, and in particular on D, we see that $u = \operatorname{Re} f$ is a solution to problem (1). Now the Cauchy integral formula allows us to write

$$f(x+iy) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z' - [x+iy]}\,dz'. \tag{2}$$

If we define

$$z^* = \frac{r^2}{\overline{z}}$$

[note that this corrects an error in the lecture, where z^* was mistakenly given as r^2/z], then z^* will be outside of C so that we will have

$$\int_C \frac{f(x'+iy')}{z' - z^*}\,dz' = 0 \tag{3}$$

Thus we may subtract this integral from (2). Now we may parameterise C as

$$z'(t) = re^{i\theta}, \qquad \theta \in [0, 2\pi];$$

let us write also $x + iy = r_0 e^{i\theta_0}$ for some θ_0. Then (2) becomes, after subtracting (3),

$$\begin{aligned}
f(x+iy) &= \frac{1}{2\pi i}\int_0^{2}\pi\left[\frac{1}{re^{i\theta} - r_0 e^{i\theta_0}} - \frac{1}{re^{i\theta} - \frac{r^2}{r_0}e^{i\theta_0}}\right] f(re^{i\theta}) ire^{i\theta}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{-\frac{r^2}{r_0}e^{i\theta_0} + r_0 e^{i\theta_0}}{\left(re^{i\theta} - r_0 e^{i\theta_0}\right)\left(re^{i\theta} - \frac{r^2}{r_0}e^{i\theta_0}\right)} re^{i\theta} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{-(r^2 - r_0^2)\frac{r}{r_0}e^{i(\theta_0+\theta)}}{\left(re^{i\theta} - r_0 e^{i\theta_0}\right)\left(re^{i\theta} - \frac{r^2}{r_0}e^{i\theta_0}\right)} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{-\frac{r^2-r_0^2}{r_0^2}e^{i(\theta_0+\theta)}}{\left(\frac{r}{r_0}e^{i\theta} - e^{i\theta_0}\right)\left(e^{i\theta} - \frac{r}{r_0}e^{i\theta_0}\right)} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\frac{r^2-r_0^2}{r_0^2}}{\left(\frac{r}{r_0} - e^{i(\theta_0-\theta)}\right)\left(\frac{r}{r_0} - e^{i(\theta-\theta_0)}\right)} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{\left|r - r_0 e^{i(\theta_0-\theta)}\right|^2} f(re^{i\theta})\,d\theta.
\end{aligned}$$

If we take the real part of this, then, since everything in the integrand is real except for f, and $\operatorname{Re} f|_{\partial D} = h$, we have

$$\begin{aligned} u(x,y) = \operatorname{Re} f &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{\left|r - r_0 e^{i(\theta_0-\theta)}\right|^2} h(r\cos\theta, r\sin\theta)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)} h(r\cos\theta, r\sin\theta)\, d\theta. \end{aligned} \tag{4}$$

As noted above, the foregoing does not actually *prove* that given a continuous function h the above function u will give a solution to problem (1); however, this can be proved by other means, though we shall not do so here. We shall however give an example.

Example. Let us start with a trivial example:

$$\Delta u = 0, \qquad u|_{\partial D} = 1.$$

Clearly the solution to this is 1. Using the integral formula (4), we have

$$\begin{aligned} u(x,y) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos\theta}\, d\theta. \end{aligned} \tag{5}$$

Now it turns out that the integrand here has an explicit antiderivative. To determine it, we work with the integral

$$\int \frac{1}{a - b\cos\theta}\, d\theta, \tag{6}$$

where we assume $a > b \geq 0$. Note that, since $\cos\theta = \cos^2\theta/2 - \sin^2\theta/2 = 2\cos^2\theta/2 - 1$,

$$a - b\cos\theta = (a+b) - b(1+\cos\theta) = (a+b) - 2b\cos^2\frac{\theta}{2},$$

so that the integral (6) may be rewritten as

$$\int \frac{1}{(a+b) - 2b\cos^2\frac{\theta}{2}}\, d\theta = \int \frac{\sec^2\frac{\theta}{2}}{(a+b)\sec^2\frac{\theta}{2} - 2b}\, d\theta.$$

Let us now make the substitution $v = \tan\theta/2$, $dv = \frac{1}{2}\sec^2\theta/2\, d\theta$; then this integral becomes, since $\sec^2 x = 1 + \tan^2 x$,

$$\begin{aligned} 2\int \frac{1}{(a+b)(1+v^2) - 2b}\, dv &= \frac{2}{a+b}\int \frac{1}{v^2 + \frac{a-b}{a+b}}\, dv \\ &= \frac{2}{a+b}\cdot\left[\frac{a+b}{a-b}\right]^{1/2} \tan^{-1}\left[\left\{\frac{a+b}{a-b}\right\}^{1/2} v\right] \\ &= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\left\{\frac{a+b}{a-b}\right\}^{1/2} v\right], \end{aligned}$$

from which we obtain finally

$$\int \frac{1}{a - b\cos\theta}\, d\theta = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\left\{\frac{a+b}{a-b}\right\}^{1/2}\tan\frac{\theta}{2}\right].$$

Now for us $a = r^2 + r_0^2$ while $b = 2rr_0$, so $a + b = (r+r_0)^2$, $a - b = (r-r_0)^2$, and $\sqrt{a^2-b^2} = r^2 - r_0^2$, and we obtain

$$\int \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos\theta}\, d\theta = 2\tan^{-1}\left[\frac{r+r_0}{r-r_0}\tan\frac{\theta}{2}\right].$$

Note that this formula is only valid on intervals which do not contain any odd integer multiple of π; for example, $(-\pi, \pi)$, $(\pi, 3\pi)$, and so on. This is because $\tan \frac{\theta}{2}$ is not defined at odd integer multiples of π. Thus to evaluate our original integral (5) we must split the interval $[0, 2\pi]$ into two pieces, $[0, \pi)$ and $(\pi, 2\pi]$, and treat each piece as an improper integral. The two integrals we thus get are

$$\frac{1}{2\pi}\int_0^{\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)}\,d\theta = \frac{1}{2\pi}\lim_{\theta\to\pi^-} 2\tan^{-1}\left[\frac{r + r_0}{r - r_0}\tan\frac{\theta}{2}\right]\Bigg|_0^{\theta} = \frac{1}{2\pi}(\pi - 0) = \frac{1}{2},$$

$$\frac{1}{2\pi}\int_{\pi}^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)}\,d\theta = \frac{1}{2\pi}\lim_{\theta\to\pi^+} 2\tan^{-1}\left[\frac{r + r_0}{r - r_0}\tan\frac{\theta}{2}\right]\Bigg|_{\theta}^{2\pi} = \frac{1}{2\pi}(0 - (-\pi)) = \frac{1}{2},$$

and finally we obtain $u = 1$, as we found originally by inspection.

hat its real part approaches h as the point (x, y) approaches the curve C *radially*, i.e., along lines directed outwards from the origin. (Note that it is *not* the case that f itself approaches h as z approaches the boundary: in fact it is not hard to show that an analytic function which is real on the boundary of a region must actually be constant throughout the region (try it!).) Let us see whether we can rewrite the real part of f as a real integral. It is convenient to consider a slight modification of this integral. The function f as defined by the above integral will also be analytic outside of the curve C. If we set

Summary:

- We give a basic introduction to the concept of *analytic continuation*, and relate it to our discussion of the logarithm.

102. Analytic continuation. Suppose that we have a function f which is analytic on some region containing a point a. Since f is analytic near a, we may expand it in a Taylor series near a and write

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)(z-a)^k, \tag{1}$$

and the series will converge to f on the largest disk centred at a on which f is analytic. (A good example to keep in mind here might be something simple like $f(z) = 1/z$, expanded around some $a \neq 0$.) Suppose that the series (1) converges on the disk D, necessarily centred at a. Now instead of starting with the function f, suppose that we instead start with the series (1); in other words, suppose that we simply given a series

$$\sum_{k=0}^{\infty} a_k (z-a)^k, \tag{2}$$

which we know converges on the disk D. Let us denote its sum by $f_1(z)$ when $z \in D$ (note that we must have $f_1(z) = f(z)$ where f is analytic). Suppose that D has radius $r > 0$, and let b be a point in D with $|b-a| = r - \epsilon$, where ϵ is small; in other words, b is close to the boundary of D. Now since $b \in D$, the series (2) must converge to an analytic function at b; we may expand thus expand f_1 in a Taylor series about b and write

$$f_1(z) = \sum_{k=0}^{\infty} b_k (z-b)^k. \tag{3}$$

This series will also converge on a disk, say D', which must have radius at least ϵ (since the disk of radius ϵ about b will be contained in the disk D by the triangle inequality). However, D' may have radius bigger than ϵ; in that case, we say that it gives an *extension* of the function f_1 to the disk D'.

Now clearly there is nothing to keep us from continuing on in this fashion, by taking further points in D or in D' and expanding the functions around those points to see whether we get further extensions. A treatment of this is given in Goursat, Chapter IV; sections 83, 84, and 86 in particular are germaine to what we shall do here, but we do not have time to go into all of the details and shall content ourselves with a particular example.

103. A particular example Let us consider the branch of the complex logarithm obtained by taking a cut along the negative real axis and requiring the angle to lie in thet set $(-\pi, \pi)$, and denote the resulting branch by L; in other words, for $\theta \in (-\pi, \pi)$ and $r \in (0, +\infty)$, we define

$$L(re^{i\theta}) = \log r + i\theta.$$

This function is analytic at $z = 1$, and thus has a power series expansion there – which we shall not give as finding it is one of the practice problems on the final review sheet; let us denote it by

$$\sum_{k=0}^{\infty} a_k (z-1)^k. \tag{4}$$

Now this series will converge to L on the largest disk centred at 1 on which L is analytic, which is easily seen to be the disk of radius 1 centred at $z = 1$, since L has a singularity at $z = 0$. Let us call this disk D_1. Since $L(re^{i \cdot 0}) \to -\infty$ as $r \to 0^+$, we see that the series (4) cannot converge on any larger disk.

Now let $b = 1 + i/2$; note that $b \in D_1$, so the series (4) converges at b. Thus we may expand the resulting function about b, obtaining

$$\sum_{k=0}^{\infty} b_k (z-b)^k. \tag{5}$$

Now of course on D_1 the series (4) converges to L, so we are really just expanding the function L around $z = b$ in this case. Thus, once again, the series will converge to L on the largest disk about b which does not include any singularity of L; we see that, as above, this will be the disk D_2 of radius $\sqrt{5/4}$ about b.

Continuing one step further, let us let $c = -1/10 + i/2$; then $|b - c|^2 = 11^2/10^2 = 121/100 < 5/4$, so $c \in D_2$. Thus we may expand the function obtained from the series (5) about $z = c$, obtaining the series

$$\sum_{k=0}^{\infty} c_k (z - c)^k; \tag{6}$$

as before we are actually expanding the function L, so we know that the series will converge to L on any disk about c on which L is analytic. This time, though, there is an extra twist. As above, clearly the largest disk on which the series (6) can converge is the disk, call it D_3, of radius $|c| = \sqrt{\frac{51}{100}}$ about c. However, clearly the point $-1/10$ satisfies $|c - (-1/10)| = 1/2 < \sqrt{\frac{51}{100}}$; in other words, $-1/10 \in D_3$. But $-1/10$ was on the branch cut of L – in other words, the function L *is not defined* at the point $-1/10$!

Let us investigate what is going on here in more detail. Let L_2 denote the branch of Log with a cut along the negative *imaginary* axis and with the angle required to lie in $(-\pi/2, 3\pi/2)$. Then clearly $L = L_2$ on the fourth, first, and second quadrants (i.e., where $\theta \in (-\pi/2, \pi)$); thus the series (4) and (5) are also the Taylor series for L_2 about $z = 1$ and $z = b$, respectively. Similarly, the series (6) is the Taylor series for L_2 about $z = c$, and will converge to L_2 on the largest disk about c on which L_2 is defined. Now geometrically it is clear that this disk is indeed the disk D_3 – in other words, the series (6) will indeed converge everywhere on D_3. It will converge to the function L_2 there. Now, as noted, $L_2 = L$ for $\theta \in (-\pi/2, \pi)$; but a little thought shows that $L_2 - L = 2\pi i$ if $\theta \in (\pi, 3\pi/2)$ – they are distinct branches over that interval.

This means, in particular, that for any $z = re^{i\theta} \in D_3$ with $\theta \in (\pi, 3\pi/2)$, the series in (6) will converge to a value equal to $L(z) + 2\pi i$. Now note that the difference of limits

$$\lim_{\theta \to \pi^-} L(re^{i\theta}) - \lim_{\theta \to -3\pi/2^+} L(re^{i\theta}) = 2\pi i$$

for any r; in other words, the function L has a jump discontinuity equal to $2\pi i$ across the branch cut – it decreases by $2\pi i$ as we cross the branch cut. But this means that if we add $2\pi i$ to L as we cross the branch cut, the result will be continuous (as long as we stay close to the branch cut, of course!) – and this is exactly what the series (6) accomplishes! In other words, given only local information about the branch L, the series was somehow able to pick out the branch L_2 which should follow L across the branch cut in order to have a function which is analytic on both sides of the branch cut.

Now we know of course that there is no branch of the logarithm which is analytic on the entire punctured plane, so the process of extension given above must run into some irresolvable obstacle at some point. Let us see what that is. Suppose that we continue expanding the function L_2 on yet another disk D_4 which is counterclockwisely further around the origin than D_3; we are evidently able to do so. If we then continue, using disks say D_5, D_6, etc., at some point we shall run into the same situation with L_2 which we had with L: the series will converge across the branch cut, but to another branch, call it L_3, on the other side of the cut, i.e., for $\theta > 3\pi/2$. As before, we will have $L_3 - L_2 = 2\pi i$ for points just across the branch cut; but $L_2 = L$ there, so $L_3 - L = 2\pi i$. Now we may take L_3 to be the branch with a cut along the positive imaginary axis and the angle θ required to lie in $(3\pi/2, 7\pi/2)$. Continuing to expand as before, we will eventually arrive back at the point $z = 1$. However, at that point we will be expanding the function L_3 instead of L, and that means that the series will converge to $L_3(1) = L(1) + 2\pi i = 2\pi i$, not $L(1) = 0$!

Note the close analogy of this procedure to what we discussed much earlier in the course about how integrating $1/z$ around the origin to get $\operatorname{Log} z$ will increase the value by $2\pi i$ – exactly the value determined here. The procedure here is however far more general, and in particular could be applied to the root functions, giving the different branches in cyclic succession as we continue expanding around the origin. (In other words, if we start with one branch of – say – the cube root function, say that which gives $1^{1/3} = 1$, and then expand it in Taylor series which circle the origin as here, then when we come back to the point 1 again we will have instead $e^{2\pi i/3}$; if we circle again, we will have $e^{4\pi i/3}$; and if we circle once more – making three times in total – we will come back to the original value, 1. With the logarithm, we would continue adding $2\pi i$ each time, which means we will never return to the original value no matter how many times we circle.) Similarly, we could apply this to the function in the last problem on the term test: in particular, the last part shows that if we were to start with the function on one side of the branch cut, and then expand it in series on disks which wrapped around the branch cut to the other side, by the time we came back to the original point there would be a difference of $-\pi/(2\sqrt{2})$.

Those who have seen – or will yet see – covering spaces should note the similarity here to the construction of the universal cover of the circle, or the punctured plane (which is homotopically equivalent, in fact there is an obvious deformation retract of the punctured plane onto the circle): we start at a certain point and then start taking curves, reducing by homotopy; since a closed loop once around the origin is not homotopic to a point, the endpoint of this curve is taken to represent a distinct point from its initial point. Similarly, a closed loop *twice* about the origin is not homotopic to either of these paths, meaning that *its* endpoint is yet another distinct point, and so on. Another way of putting all of this together is that, should we define the logarithm on the universal cover of the punctured plane instead of the punctured plane itself, it would become a single-valued analytic function. Taking a branch would then correspond to restricting the domain to some piece of this universal cover for which the covering map is a homeomorphism onto a cut plane. (What we just noted about root functions shows that something similar is true for them, but instead of needing to use the universal cover of the punctured plane, we need to use, for the nth-root function, the n-sheeted cover.) Similarly, functions with more complicated branch points – such as the function on the term test, which has four branch points; or the arctangent function, which has two – can be defined on covers of the multiply-punctured plane. The fact that if we traverse a loop around all four points we come back to the same value means however that we do not get the universal cover of the quadruply-punctured plane in this case, but rather some other set, an elucidation of which is however beyond the knowledge of the present author, who will therefore retire before he says anything more wrong than he already has.

104. A specific example Let us consider the complex logarithm, and try to find its Taylor series expansion about $z = 1$; in particular, let us see if we can determine what the radius of convergence of that power series must be. Immediately there is a problem: shouldn't we have to specify which branch of the logarithm we are using? – after all, the logarithm itself is a multivalued function but power series always give single-valued functions! On the other hand, what do we need to compute the Taylor series of a function at $z = 1$? Let us let f denote the 'logarithm' (whatever that means in the end, e.g., a particular branch or whatever we end up deciding on); then what we need is

$$f(1), \quad f'(1), \quad f''(1), \cdots.$$

Now $f(1)$ will depend on the branch, but since all of the different branches differ by only a constant value, all branches defined at $z = 1$ will have equal values for the higher derivatives! Moreover, even though $f(1)$ depends on the branch, we still know that no matter which branch we choose, as long as it is defined at $z = 1$, we will have $f(1) = 2n\pi$ for some $n \in \mathbf{Z}$. Thus, there is a sequence of numbers $a_1, a_2, \cdots$ such that for any branch of Log defined at $z = 1$, there will be some $n \in \mathbf{Z}$ such that this branch equals

$$2n\pi + \sum_{k=1}^{\infty} a_k (z-1)^k \tag{7}$$

(where the sequence a_k is given, of course, by

$$a_k = \frac{1}{k!} \frac{d^k}{dz}^k \operatorname{Log} z \Bigg|_{z=1},$$

though we are not too interested in this fact right here). Now whether the series (7) converges for a particular value of z is clearly independent of the value of n; thus in determining its radius of convergence we may use any branch we like. Now from what we have seen in lecture, the series (7) will converge on the largest disk centred at $z = 1$ on which the branch we are dealing with is analytic. Since all branches of the logarithm have a singularity at the origin, the largest possible radius of convergence for the series (7) is clearly 1. That this is in fact actually its radius of convergence can be seen by taking the branch of the logarithm with a cut along the negative real axis and an angle required to lie in $(-\pi, \pi)$: clearly the origin is the nearest point on the cut to $z = 1$, and since this branch will be analytic everywhere else, its Taylor series (7) (here $n = 0$, but as noted above this is not important) will converge on the disk of radius 1 centred at $z = 1$, call it D. This is then the same for all other branches of the log function which are defined at $z = 1$.

But now it seems that we have a problem! Suppose that we take a branch cut along a line radially outwards from the origin and making a very small angle α (positive or negative) with the positive real axis: clearly the resulting function is defined at $z = 1$, so by the foregoing, its Taylor series at $z = 1$ must be given by (7) for some $n \in \mathbf{Z}$, and must therefore converge on D. But as long as the angle α is between $-\pi/2$ and $\pi/2$, the branch cut we have chosen will clearly intersect D, meaning that the branch is *not* analytic everywhere on D. So evidently we have a case of a Taylor series which converges even across a singularity of the function it is suppose to represent!

With a little more thought, though, this is actually not that confusing. Let us pick some specific numbers to make things concrete. Thus let L_1 denote the branch of Log obtained by making a cut along the line $\theta = \pi/4$ and requiring the angle to lie in the set $(-7\pi/4, \pi/4)$, and let L_2 denote the branch of Log obtained by making a cut instead along the line $\theta = \pi/2$ and requiring the angle to lie in $(-3\pi/2, \pi/2)$. Then clearly both of these branches are defined at $z = 1$, and $L_1(1) = L_2(1) = 0$, so that they have the same Taylor series, given by (7) with $n = 0$. Moreover, if we write out the definitions of these two functions more carefully, we see that for $\theta \in (-3\pi/2, \pi/4)$ we have

$$L_1(re^{i\theta}) = L_2(re^{i\theta}) = \log r + i\theta,$$

while for $\theta \in (-7\pi/4, -3\pi/2)$ we have

$$L_1(re^{i\theta}) = \log r + i\theta$$

but, since the corresponding angle in $(-3\pi/2, \pi/2)$ is $2\pi + \theta$, we have

$$L_2(re^{i\theta}) = \log r + i(2\pi + \theta);$$

in other words, the difference is given by (for $\theta \in (-7\pi/4, \pi/4)$, $\theta \neq -3\pi/2$)

$$L_1(re^{i\theta}) - L_2(re^{i\theta}) = \begin{cases} 0, & \theta \in (-3\pi/2, \pi/4) \\ -2\pi i, & \theta \in (-7\pi/4, -3\pi/2). \end{cases}$$

Now since the branch cut for L_2 does not intersect the disk D, the series (7) will not only converge on D, it will actually converge to $L_2(z)$ everywhere on D. Thus we can see that the series (7) will converge to $L_1(z)$ as long as the argument of z lies between $-\pi/2$ and $\pi/4$, while it will converge to $L_1(z) + 2\pi i$ if the argument of z is greater than $\pi/4$ (in both cases, of course, we assume that $z \in D$ as otherwise (7) does not converge at all).

Summary:

- We give a more careful treatment of the method of solving Laplace's equation using linear solutions.

105. Linear solutions to boundary value problems for Laplace's equation. Basically, when we are solving a boundary-value problem, we are trying to find, by whatever method, a function u which satisfies two conditions: (i) it must satisfy Laplace's equation, i.e., its Laplacian must vanish; (ii) its value on the boundary of the region must equal the specified function. 'By whatever method' means that we do not care how we obtained the function, only that it satisfies these two conditions; in other words, the method does not need to be constructive or computational in any way. (This is all right since there are general theorems which guarantee that the solutions to problems like this are unique, at least when the boundary data is sufficiently nice.) Obviously, then, one way of 'finding' the solution would be to try one function after another until (hopefully) the correct one is found. Since there are infinitely many different functions we might need to try, though, that isn't a very practical idea. On the other hand, most functions one could think of writing down will certainly not satisfy condition (i); for example, $\sin x$ certainly doesn't, nor does $\sin x \sin y$, etc.. So maybe a good starting point would be to try to find a collection of functions (ideally all functions, though in practice that is again too many) which satisfy Laplace's equation, and then see if maybe we can somehow find one of them which also has the correct values on the boundary. Now one especially simple class of functions which satisfy Laplace's equation are the linear functions, $g(x, y) = a + bx + cy$: this is because we can calculate:

$$\frac{\partial g}{\partial x} = a, \qquad \frac{\partial g}{\partial y} = b,$$
$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x} a = 0, \qquad \frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y} b = 0,$$

so

$$\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0.$$

Now that we know that these functions all satisfy Laplace's equation, if we are trying to solve a boundary-value problem we only need to find *numbers* a, b, and c such that the boundary conditions are satisfied. In other words, we are going to substitute the linear solution $a + bx + cy$ into the boundary conditions in (ii) and try to solve for the numbers a, b, and c. (For anything other than very special boundary conditions, of course, this will not be possible, because the linear solutions are too special; but for the problems on this assignment this is possible at some point.) Generally we solve for the numbers a, b and c either 'by inspection' or by substituting in values for x and y to obtain a system of equations that they must satisfy, which we then try to solve.

Consider the following trivial examples on the unit square $D = \{(x, y) \,|\, x, y \in [0, 1]\}$:

Example 1. Solve the following problem:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = 1.$$

If we try a linear solution in this case, then we would say, let us see whether a solution of the form $u = a + bx + cy$ can solve this problem. It clearly solves $\Delta u = 0$; now the condition $u|_{\partial D} = 1$ means $a + bx + cy = 1$ whenever $(x, y) \in \partial D$. Thus, for example, if we let $x = 1$, $y = 0$ (which is clearly a point in ∂D), we get $a + b = 1$; if we let $x = 1$, $y = 1$, we get $a + b + c = 1$, which means $c = 0$; if we let $x = 0$, $y = 0$, we get $a = 1$, which also means that $b = 0$ since $a + b = 1$. Thus the only linear function which could possibly satisfy the boundary conditions is $u = 1$. But this function clearly does actually satisfy the boundary condition on all of ∂D, and since it satisfies Laplace's equation (as it had to since it was a particular example of the class of linear functions, each one of which is a solution to Laplace's equation), it must be the desired solution.

Example 2. Solve the following problem:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = x.$$

Let us again try a linear solution: thus we wish to determine whether there are constants a, b, and c such that $u = a + bx + cy$ (which must satisfy Laplace's equation) satisfies the boundary condition. In this case things get a bit more interesting. Suppose that $y = 0$; then we have, for all $x \in [0, 1]$, that $a + bx = x$. If we set $x = 0$, this gives $a = 0$, so $bx = x$; if we set $x = 1$ (or, for that matter, if we let x be any nonzero number), this gives $b = 1$. So far, then, we know that we must have $u = x + cy$. Now if $x = 0$ and $y \neq 0$, say $y = 1$, then we have by the boundary condition that $cy = c = 0$. Thus the only linear solution that could possibly satisfy the boundary conditions is $u = x$. Now this does actually clearly satisfy the boundary conditions; and since it also satisfies Laplace's equation, it must be the desired solution.

Note however that the solution cannot always be read off from the boundary conditions as in the two examples above. For example, the boundary condition in example 2 could have been expressed as follows: $u|_{\partial D} = 0$, $x = 0$, 1, $x = 1$, x, $y = 0$ or $y = 1$, which we might not immediately recognise – but the method above would give $u = x$ as the only solution regardless. For a more involved example, consider the following problem:

Example 3. Solve the following problem:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = \begin{cases} x, & y = 0 \\ x + 1, & y = 1 \\ y, & x = 0 \\ y + 1, & x = 1 \end{cases}$$

Let us see whether we can find a linear solution; thus suppose that $u = a + bx + cy$. The first of the boundary conditions gives $a + bx = x$ for $x \in [0, 1]$; if $x = 0$ this gives $a = 0$, while if $x = 1$ this gives $b = 1$. Thus we already know that if there is such a solution, it must be of the form $u = x + cy$. Now consider the third of the boundary conditions ($u = y$ when $x = 0$); this gives $cy = y$, which, setting $y = 1$, gives $c = 1$. Thus the only possible solution would be $u = x + y$. But now we need to show that this does indeed satisfy the other two boundary conditions. At $y = 1$ this expression gives $u(x, 1) = x + 1$, which is the correct expression; and at $x = 1$ it gives $u(1, y) = 1 + y = y + 1$, which is again the correct expression. Thus $u = x + y$ satisfies the boundary conditions, and since it satisfies Laplace's equation, it must be the desired solution.

These are of course rather simple examples, but they demonstrate the technique. More complicated examples were given in the lecture notes (see August 11, Section 38, pp. 3 – 4, August 13, Section 40, p. 4); it may be helpful to restudy these in the light of the above explanations.

The same technique is applicable to the problems on the last homework assignment: we posit that u takes a certain form, and then determine the coefficients by matching that form to the given boundary conditions.