Summary:
- We give a description of the complex number system.
- We then give a description of the complex plane and indicate why it is something which might be useful.

I. INTRODUCTION TO THE COMPLEX PLANE

1. Complex numbers. We probably saw complex numbers for the first time when we learned how to solve quadratic equations. For example, the equation

$$x^2 = -1$$

has no solution over the real numbers. It turns out to be useful in algebra, and even more in analysis, to *extend* our number system by including an extra quantity, written i, which behaves exactly like a real number except that it has the property

$$i^2 = -1. \tag{1}$$

A general number in our new number system can be written in the form $a + bi$, where a and b are arbitrary *real* numbers,[1] and we require that these numbers satisfy all of the standard rules of algebra, augmented by equation (1). Thus, for example, the product of two complex numbers is given by

$$(a + bi)(c + di) = ac + bi \cdot c + a \cdot di + bi \cdot di = ac + bci + adi + bdi^2 = ac - bd + (bc + ad)i.$$

(As shown here, whenever we write out a complex number we always combine real and imaginary terms when possible.)

We generally use the letters z and w to denote complex numbers, and x and y to denote real numbers. We let $\mathbf{C}$ denote the set of all complex numbers. If $z = a + bi$ is a complex number, we call a the *real part* of z and b the *imaginary part* of z, and write $a = \operatorname{Re} z$, $b = \operatorname{Im} z$. Two complex numbers $a + bi$ and $c + di$ are *equal* if and only if their real and imaginary parts are equal.[2]

To every complex number $a + bi$ there corresponds another complex number known as its *conjugate* and given by $a - bi$.[3] If z is any complex number, we write $\overline{z}$ for its conjugate. The conjugate will be seen later to have many uses, but for the moment we note its use in finding inverses. First, note that if $z = a + bi$, then

$$z\overline{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

Thus if $a + bi \neq 0$, then

$$\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2},$$

which is defined since $a + bi \neq 0$ implies that at least one of a and b is nonzero, so $a^2 + b^2 > 0$. This is the desired formula for the inverse of a complex number.

See Goursat, §1.

2. The complex plane. If z is any complex number, it determines two real numbers $\operatorname{Re} z$ and $\operatorname{Im} z$, and is in turn uniquely determined by these two numbers. This suggests that, just as we may think of arbitrary real numbers as points on the *real number line*, we may think of arbitrary complex numbers as points in the *complex plane*. Specifically, given a plane with perpendicular axes which we call x and y, we associate with any complex number z the point in this plane whose x-coordinate is $\operatorname{Re} z$ and whose y-coordinate is $\operatorname{Im} z$. While complex numbers are *per se* abstract objects without any direct concrete significance, this association allows us to think and speak of them as points in the plane. We shall do this whenever it seems

[1] Whenever we write an arbitrary complex number as $a + bi$, it will always be assumed that a and b are real.

[2] This means that the set $\{1, i\}$ is a basis for $\mathbf{C}$ considered as a real vector space.

[3] We remind the reader that the presence of a $+$ or $-$ in front of a quantity does not guarantee the resulting sign; in other words, $+b$ can be negative and $-b$ can be positive, and both will be respectively when b is negative.

convenient; thus we shall speak of "the point $a+bi$", etc., when more carefully we should say "the point corresponding to the complex number $a+bi$".

Given the foregoing, it is clear that the point corresponding to the conjugate of a complex number $a+bi$ is simply the reflection in the x-axis of the point corresponding to $a+bi$.

The foregoing connection between complex numbers and points in a plane, while it may be interesting, would not be particularly useful if the *geometric* properties inherent in the Euclidean plane were not somehow related to *algebraic* or *analytic* properties of the complex numbers its points represent. We shall see throughout this course that there are in fact many and deep connections between the geometry of the plane on the one hand and the algebraic and analytic properties of complex numbers on the other. Here we shall indicate one example.

EXAMPLES. One simple example is as follows. Suppose that $z = a+bi$ and $w = c+di$ are any two complex numbers. Then clearly

$$\overline{z}w = (a-bi)(c+di) = ac+bd+i(ad-bc).$$

Now if we think of the vectors (corresponding to the points) corresponding to $a+bi$ and $c+di$, i.e., $\mathbf{v} = a\mathbf{i}+b\mathbf{j}$, $\mathbf{u} = c\mathbf{i}+d\mathbf{j}$, we see that their dot product is $\mathbf{v}\bullet\mathbf{u} = ac+bd$ while their cross product is $\mathbf{v}\times\mathbf{u} = (ad-bc)\mathbf{k}$; in other words, roughly, the real part of $\overline{z}w$ is the dot product of the vectors corresponding to z and w, while the imaginary part is their cross product.[4] We shall see some other relations of this sort when we talk about derivatives of functions of a complex variable; it turns out that, when viewed as a vector field, the derivative of the conjugate of such a function essentially encodes the divergence and curl of the vector field.[5]

As another, more interesting, example, let $a+bi$ be any complex number, and consider the corresponding point in the plane. This point has polar coordinates (r,θ), where r is the distance from the origin to the point and θ is the angle from the positive x-axis to the ray from the origin passing through the point. In symbols, this becomes

$$r = \sqrt{a^2+b^2}, \qquad \cos\theta = \frac{a}{\sqrt{a^2+b^2}}, \qquad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$
$$a = r\cos\theta, b = r\sin\theta$$

Note that θ is only defined up to a multiple of 2π: the two polar coordinate expressions (r,θ) and $(r,\theta+2\pi)$ determine exactly the same point in the plane. We shall see shortly that for many important functions to be continuous (in an appropriate sense) on the complex plane, there is no way around this ambiguity: it is simply something which must be dealt with.

Now suppose that $c+di$ is any other complex number which satisfies $c^2+d^2 = 1$: this means that the point corresponding to $c+di$ lies on the unit circle. If we let (r_0,θ_0) denote the polar coordinates of this point, then we have $r_0 = 1$, while θ_0 satisfies $\cos\theta_0 = c$, $\sin\theta_0 = d$.[6] Now applying basic trigonometric identities, we obtain

$$\begin{aligned}(a+bi)(c+di) &= ac-bd+i(ad+bc)\\ &= r\cos\theta\cos\theta_0 - r\sin\theta\sin\theta_0 + i(r\cos\theta\sin\theta_0 + r\sin\theta\cos\theta_0)\\ &= r\cos(\theta+\theta_0) + ir\sin(\theta+\theta_0)\\ &= r\left[\cos(\theta+\theta_0) + i\sin(\theta+\theta_0)\right],\end{aligned}$$

from which it is evident that the point corresponding to the product $(a+bi)(c+di)$ is simply that corresponding to $a+bi$ rotated counterclockwise by the angle θ_0!

[4] It turns out that there is a *four-dimensional* extension of the real numbers called the *quaternions*, which contain the complex numbers, and which in some sense generalises results of this sort to full three-dimensional vectors. We shall not deal with these in this course, though, except for a few asides like this one.

[5] While interesting, these examples are somewhat tangential to the main content of this course.

[6] If you are familiar with De Moivre's theorem, it is useful to note that this means that $c+di = \cos\theta_0 + i\sin\theta_0$.

Summary:
- We discuss another geometric interpretation of complex multiplication.
- We then discuss taking powers and roots of complex numbers, and the geometric interpretation of these operations.

We have just observed that multiplying a complex number by another complex number of unit modulus is equivalent to rotating the original complex number by an angle equal to that of the second complex number. It turns out that multiplication by a general complex number can be viewed as the composition of a rotation and an isotropic scaling. Let us see how this works. Suppose that we have two complex numbers,

$$z = r(\cos\theta + i\sin\theta), \qquad w = r'(\cos\theta' + i\sin\theta').$$

Then their product comes out to be (the angular part is exactly analogous to what we saw at the end of the notes of May 5)

$$\begin{aligned} zw &= rr'(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta') \\ &= rr'(\cos\theta\cos\theta' - \sin\theta\sin\theta' + i[\sin\theta\cos\theta' + \cos\theta\sin\theta']) \\ &= rr'[\cos(\theta+\theta') + i\sin(\theta+\theta')]; \end{aligned} \tag{1}$$

in other words, the point corresponding to zw is exactly the point corresponding to z, rotated by θ' and scaled by r'. This is the sense in which multiplication by a complex number is just a rotation and a scaling. (This is related to some of the problems on the review sheet!)

3. Exponentiation. We have seen that the affect of multiplication on the angular part of a complex number is just a rotation. What happens under exponentiation? Let $z = r(\cos\theta + i\sin\theta)$; then we see that, by the formula in (1) above,

$$\begin{aligned} z^2 &= z\cdot z = r^2(\cos 2\theta + i\sin 2\theta), \\ z^3 &= z\cdot z^2 = r(\cos\theta + i\sin\theta)\cdot r^2(\cos 2\theta + i\sin 2\theta) = r^3(\cos 3\theta + i\sin 3\theta), \end{aligned}$$

and so on, so that it is evident that for any positive integer m we have

$$z^m = r^m(\cos m\theta + i\sin m\theta).$$

To try to get some sense of what this means geometrically, let us first consider the case $r = 1$; then $r^m = 1$ for all m and we have simply

$$z^m = \cos m\theta + i\sin m\theta.$$

Now any complex number of unit modulus is represented in the complex plane by a point on the unit circle, and completely determined by the angle between a ray drawn from the origin to that point and the positive x-axis, measured in a counterclockwise direction: this is just the number θ above. This formula then tells us that the point corresponding to z^m is also on the unit circle, but with an angle from the positive x-axis equal to m times that of the point corresponding to z. In other words, if we must traverse an angle θ to arrive at z, we must traverse an angle of $m\theta$ to arrive at z^m.

Suppose now that we consider the affect of exponentiation on not just a single point on the unit circle but rather an *arc*, say from $\theta = 0$ to $\theta = \theta_0$ for some $\theta_0 > 0$. The point corresponding to θ_0, namely $\cos\theta_0 + i\sin\theta_0$, will be mapped by this exponentiation to $\cos m\theta_0 + i\sin m\theta_0$; and it is clear that every point with $\theta \in [0, \theta_0]$ will be mapped to a point with $\theta \in [0, m\theta_0]$. Thus exponentiation simply stretches out the original arc.

With this in mind, let us consider the affect of exponentiation on an angular wedge, namely on the set of all points (of whatever modulus) whose angle with the positive x-axis lies between 0 and θ_0. Such a point can be written in the form $z = r(\cos\theta + i\sin\theta)$, where $\theta \in [0, \theta_0]$, and $z^m = r^m(\cos m\theta + i\sin m\theta)$; from the foregoing, then, it is clear that this point will lie inside a 'wedge' (it may have an angle greater than π and hence not really be a proper 'wedge' anymore) extending from 0 to $m\theta_0$.

Now there is no particular reason to restrict the lower angular bound on the wedge to be 0; we may as well consider a wedge $[\theta_1, \theta_2]$. The same logic shows that this will be mapped to a wedge $[m\theta_1, m\theta_2]$.

In particular, if we consider the wedge from 0 to π and let $m = 2$, we see that the image under exponentiation is the 'wedge' from 0 to 2π, i.e., the entire complex plane. The same is true if we consider the wedge from 0 to $\frac{2\pi}{3}$ and let $m = 3$, and in general, if m is any positive integer, then the wedge from 0 to $\frac{2\pi}{m}$ will be mapped to the entire complex plane by the map $z \mapsto z^m$. Similarly, the wedge from $\frac{2\pi}{m}$ to $\frac{4\pi}{m}$ will also be mapped to the entire complex plane, and so will the wedges from $\frac{2n\pi}{m}$ to $\frac{2(n+1)\pi}{m}$ for any $n = 0, 1, \ldots, m-1$.

While we do not quite have all of the necessary tools to make the following picture precise, it provides much useful intuition and I think is simple enough to understand. We may think of exponentiation by a positive integer as an endpoint in a process that starts with exponentiation by 1 (i.e., doing nothing!) and then slowly increases the exponent *through all real numbers* until it reaches m. Under this kind of a map, the wedge from 0 to $\frac{2\pi}{m}$ (say) will be slowly stretched out (with the bottom edge, i.e., that along the x-axis, remaining fixed) until the outer edge finally reaches the x-axis. Under the same map, the wedge from $\frac{2\pi}{m}$ to $\frac{4\pi}{m}$ will behave slightly differently: the lower edge $\frac{2\pi}{m}$ also moves until it reaches the positive x-axis, while the upper edge $\frac{4\pi}{m}$ moves even faster so that by that point it has travelled one full 2π past the positive x-axis. Similar things can be said about the additional wedges.

What all of this means is that under exponentiation by a positive integer, the wedges $\frac{2\pi n}{m}$ to $\frac{2\pi(n+1)}{m}$ are each rotated and stretched in such a way as to cover the entire complex plane exactly once.[1] This means that each complex number is the image under the exponentiation map of exactly one point from each of these wedges. A little thought shows that this means that each complex number (except 0) has exactly m mth roots.

More precisely, suppose that $z = r(\cos\theta + i\sin\theta)$ is some complex number. Now for each positive real number r there is exactly one positive real number R satisfying $R^m = r$, and we denote this unique positive real mth root by $r^{\frac{1}{m}}$. Given this, for $n = 0, 1, \ldots, m-1$, let $w_n = r^{\frac{1}{m}}\left(\cos\frac{\theta+2\pi n}{m} + i\sin\frac{\theta+2\pi n}{m}\right)$; then clearly

$$\begin{aligned} w_n^m &= \left(r^{\frac{1}{m}}\right)^m (\cos(\theta + 2\pi n) + i\sin(\theta + 2\pi n)) \\ &= r(\cos\theta + i\sin\theta) = z, \end{aligned}$$

so that each of the w_n is an mth root of z. More specifically, if we assume that $\theta \in [0, 2\pi]$, then it is clear that w_n is in the nth of the above wedges. We note that $w_m = w_0$, and in general $w_{n+km} = w_n$ for any positive integer k. It can be shown that the w_n are the only complex mth roots of z, and that z therefore has exactly m distinct mth roots, as claimed. [The proof is not that hard: suppose that $w = r'(\cos\theta' + i\sin\theta')$ is any mth root of z, i.e., that $w^m = z$; this means that

$$r'^m(\cos m\theta' + i\sin m\theta') = r(\cos\theta + i\sin\theta),$$

which means that $r'^m = r$, i.e., $r' = r^{\frac{1}{m}}$, and that there is an integer n such that $m\theta' = \theta + 2n\pi$, which gives $\theta' = \frac{\theta}{m} + \frac{2n\pi}{m}$ for some integer n. Now dividing n by m we can find integers q and r such that $n = qm + r$ and $r \in \{0, 1, 2, \ldots, m-1\}$; thus $\theta' = \frac{\theta}{m} + \frac{2(qm+r)\pi}{m} = \frac{\theta}{m} + 2q\pi + \frac{2r\pi}{m}$ and this w is equal to w_r.]

4. Complex derivatives. Cauchy-Riemann equations In first-year calculus we learned that the derivative of a real-valued function of a single real variable, if it exists, is given by the limit

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}.$$

In multivariable calculus, we learned about taking *partial derivatives*, which are derivatives in a single direction at a time; we couldn't take the derivative 'with respect to a vector' since we had no way of dividing by a vector.[2] Those of you who have seen how derivatives of functions from $\mathbf{R}^n$ to $\mathbf{R}^m$ can be viewed as linear operators between those spaces will still recall that the components of the matrix representations of those operators are still calculated as partial derivatives, i.e., even in that case we reduce back to the case of a single function of a single variable.

[1] Well, *almost* exactly once. To be precise we should only include one of the two edges, restricting the angle to lie in a half-open interval.

[2] This is not *entirely* correct and there is in fact a nice way in which the gradient can be viewed as a derivative $\frac{df}{d\mathbf{r}}$. But that is probably more of a notational shorthand than anything fundamental, unlike what we are about to do with complex numbers.

In complex analysis, though, we can go further since we have a well-defined way of dividing by complex numbers even though they are two-dimensional quantities (at least over $\mathbf{R}$!). Let $f : \mathbf{C} \to \mathbf{C}$ be a complex-valued function of a complex variable, and consider the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$

where now h is allowed to be a *complex* number. Since h is complex, this means that we are taking a two-dimensional limit. As we have learned in multivariable calculus, a two-dimensional limit can only exist if directional limits from different directions exist and are equal (and it may fail to exist even then). Let us consider what the above limit looks like in the two cases where we restrict h to go to zero along the real and imaginary numbers (in terms of the complex plane, this means that h goes to zero along the horizontal and vertical axes, respectively). First, let us write out f explicitly in terms of its real and imaginary parts as (writing $z = x + iy$)

$$f(x+iy) = P(x,y) + iQ(x,y),$$

and assume that all partial derivatives $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ exist. If $h = \Delta x$ is real, the quotient inside the limit becomes

$$\begin{aligned} \frac{f(x+iy+\Delta x) - f(x+iy)}{\Delta x} &= \frac{P(x+\Delta x, y) + iQ(x+\Delta x, y) - [P(x,y) + iQ(x,y)]}{\Delta x} \\ &= \frac{[P(x+\Delta x, y) - P(x,y)] + i[Q(x+\Delta x, y) - Q(x,y)]}{\Delta x}. \end{aligned}$$

Since the partial derivatives $\frac{\partial P}{\partial x}$ and $\frac{\partial Q}{\partial x}$ exist, in the limit as Δx goes to zero this becomes

$$\frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}.$$

This gives the original (two-dimensional) limit along the real axis. To find the limit along the imaginary axis, let $h = i\Delta y$ (note the i!); then we obtain

$$\begin{aligned} \frac{f(x+iy+i\Delta y) - f(x+iy)}{i\Delta y} &= \frac{P(x, y+\Delta y) + iQ(x, y+\Delta y) - [P(x,y) + iQ(x,y)]}{i\Delta y} \\ &= -i\left\{\frac{[P(x, y+\Delta y) - P(x,y)] + i[Q(x, y+\Delta y) - Q(x,y)]}{\Delta y}\right\}, \end{aligned}$$

so that since the partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial y}$ exist this becomes

$$-i\left\{\frac{\partial P}{\partial y} + i\frac{\partial Q}{\partial y}\right\} = \frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y}.$$

For the full two-dimensional limit to exist, this must equal the limit along the real axis; thus we must have

$$\frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x},$$

which gives the celebrated *Cauchy-Riemann equations*

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

To sum up: for a function f of a complex variable to have a derivative at a point, its real and imaginary components P and Q must have partial derivatives at that point and those partial derivatives must satisfy the Cauchy-Riemann equations. It can be shown (see Goursat, §3) that if the partial derivatives of P and Q are also continuous at the point in question, then these conditions are sufficient in that f is then guaranteed to have a derivative at that point. Functions whose real and imaginary parts satisfy the Cauchy-Riemann equations but which do not have a derivative shall not concern us much in this course.

When f has a derivative at a certain point, by the foregoing that derivative is given by either of the expressions

$$f'(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y}.$$

Other equivalent expressions can also be derived; see Goursat, §3, equation (2).

Let us consider a specific example of the foregoing.

EXAMPLES. Let us consider a very simple function:

$$f(z) = z^2.$$

To find its real and imaginary parts, let $z = x + iy$; then

$$f(z) = f(x+iy) = (x+iy)^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + i(2xy),$$

whence we see that its real and imaginary parts are, respectively,

$$P(x,y) = x^2 - y^2, \qquad Q(x,y) = 2xy.$$

We leave it as a worthwhile exercise to the reader to show that these do in fact satisfy the Cauchy-Riemann equations. Since they certainly have continuous partial derivatives, we see that f must have a derivative at any point z. The formulas above give this derivative as

$$f'(z) = f'(x+iy) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = 2x + 2iy = 2z.$$

(This should not be a surprise, since we know from real-variable calculus that the derivative of x^2 is $2x$.) In this case, we can also derive this result directly, as follows:

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h\to 0} \frac{z^2 + 2zh + h^2 - z^2}{h} = \lim_{h\to 0} \frac{2zh + h^2}{h} = \lim_{h\to 0} (2z+h) = 2z. \end{aligned}$$

This result turns out to be typical: most of the standard functions we are familiar with from calculus which have derivatives as functions of a real variable *also* have derivatives as functions of a *complex* variable, and the derivatives are the same. (There is a very good reason for this, which will become clearer throughout the course: it is tied up with the fact that most of the functions we deal with in calculus do not just have a single derivative but are rather *real analytic*, i.e., are equal to their Taylor series expansions. Such functions always extend to differentiable functions of a complex variable, and this is one of the major links from real to complex variable theory.)

As a still elementary but slightly more complicated example, let us show that the power rule of elementary calculus holds for functions of a complex variable, if we restrict ourselves to positive integer exponents. (It holds for more general exponents, too, at least away from $z = 0$, but that will require a separate treatment.) Thus let m be a positive integer, and define $f(z) = z^m$. Then we have

$$\begin{aligned} \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \frac{(z+h)^m - z^m}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=0}^{m} \binom{m}{k} z^{m-k}h^k - z^m\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(mz^{m-1}h + \frac{m(m-1)}{2}z^{m-2}h^2 + \cdots\right) \\ &= \lim_{h\to 0} \left(mz^{m-1} + \frac{m(m-1)}{2}z^{m-2}h + \cdots\right) = mz^{m-1}, \end{aligned}$$

since all terms in $\cdots$ have at least an h^2 in them and hence must go to zero as h does. Thus we have $f'(z) = mz^{m-1}$, exactly as in the real-variable case.

Summary:
- We wrap up some loose ends from last time.
- We discuss how differentiation rules from elementary calculus can be extended to the current setting.
- We discuss multiple-valued functions and give a brief introduction to the notion of *branch cut.*

5. Harmonic functions. If a function $f'(z)$ has a derivative throughout a region, we say that it is *analytic* in that region.[1] From last time, we know that if we write f as

$$f(x+iy) = P(x,y) + iQ(x,y),$$

then, assuming that P and Q possess continuous first-order partial derivatives, f will be analytic if P and Q satisfy the Cauchy-Riemann equations

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

It turns out that these equations impose a very strong condition on P and Q, namely that they be *harmonic*, i.e., that they satisfy Laplace's equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Assuming that P and Q possess continuous second-order partial derivatives, this can be shown easily as follows:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial Q}{\partial y} + \frac{\partial}{\partial y}\left[-\frac{\partial Q}{\partial x}\right] = \frac{\partial^2 Q}{\partial x \partial y} - \frac{\partial^2 Q}{\partial y \partial x} = 0,$$

since under the above assumption the mixed partial derivatives of Q commute. The calculation for Q is similar and we leave it to the reader as an exercise.

To summarise, then, we have the implication

$$f \text{ analytic} \Longrightarrow \operatorname{Re} f, \operatorname{Im} f \text{ harmonic}.$$

Note that the reverse implication is false, since if P and Q are two harmonic functions there is in general no reason at all to expect them to satisfy the Cauchy-Riemann equations. Note also that for us the term *harmonic* is applied only to real-valued functions of real variables; we do not speak of a function f of a complex variable being harmonic. (We could define *analytic* for functions of a real variable – it is simply that the function have a convergent power series representation – but we have not done so as we shall have no particular need for this concept by itself.)

Harmonic functions are very important in many areas of physics and science, as they can be used to describe temperature distributions, static electric fields, and steady-state fluid flows, for example. We shall see later that one major application of complex variable theory lies in the use of analytic functions *qua* conformal maps to find solutions to Laplace's equation in nontrivial geometries.

Given a harmonic function P, there is a harmonic function Q, unique up to an additive constant, such that $f(x+iy) = P(x,y) + iQ(x,y)$ is analytic. This is discussed in Goursat, §3, and also in §9 below.

6. Differentiation rules. We have already seen one example (at the end of §4 from last time) where a differentiation rule from elementary calculus carried across essentially unchanged to the current setting. It turns out that almost all of the differentiation rules from elementary calculus do also carry over to functions of a complex variable: for example, the product rule and quotient rule do, since the proofs of those two

[1] The word *analytic*, when applied to a real-valued function of a real variable, means that the function can be extended in a power series, i.e., that the Taylor series of the function converges to the function on some interval. We shall show later that, for functions of a complex variable, existence of the derivative throughout an appropriate region allows us to conclude that the function has derivatives of all orders, and that the Taylor series about each point converges to the function on some disc. Thus our terminology is consistent with the real-variable case.

rules work equally well for complex independent variables as they do for real. This means that derivatives of rational functions (quotients of polynomials) can be found exactly as for functions of a real variable.

The chain rule also carries over to the current setting, as can be seen as follows. Suppose that f and g are analytic functions, and let $z \in \operatorname{dom} f$ be such that $f(z) \in \operatorname{dom} g$. Then since f and g are analytic we have

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f'(z), \qquad \lim_{h'\to 0} \frac{g(f(z)+h')-g(f(z))}{h'} = g'(f(z)).$$

Now the first relation can be rewritten in the following way:

$$\lim_{h\to 0} \frac{f(z+h)-f(z)-f'(z)h}{h} = 0.$$

Let us write $\epsilon(h) = f(z+h) - f(z) - f'(z)h$, so that this result becomes $\lim_{h\to 0} \frac{\epsilon(h)}{h} = 0$. Similarly let us write $\epsilon'(h') = g(f(z)+h') - g(f(z)) - g'(f(z))h'$.[2] Then we note that

$$\begin{aligned}
\lim_{h\to 0} \frac{g(f(z+h))-g(f(z))}{h} &= \lim_{h\to 0} \frac{g(f(z)+f'(z)h+\epsilon(h))-g(f(z))}{h} \\
&= \lim_{h\to 0} \frac{g'(f(z))[f'(z)h+\epsilon(h)]+\epsilon'(f'(z)h+\epsilon(h))}{h} \\
&= g'(f(z))f'(z) + \lim_{h\to 0}\left[g'(f(z))\frac{\epsilon(h)}{h} + \frac{\epsilon'(f'(z)h+\epsilon(h))}{h}\right];
\end{aligned}$$

but the limit of the first fraction is zero by what we know about $\epsilon(h)$, while the limit of the second is also zero by what we know about $\epsilon(h)$ and $\epsilon'(h)$. Thus we have

$$\frac{d}{dz} g(f(z)) = g'(f(z))f'(z),$$

exactly as we do in elementary calculus.

We shall see shortly that, given appropriate extensions of the elementary transcendental functions of calculus (the trigonometric, exponential, and logarithmic functions), the derivatives of all of these functions are also what one would expect from calculus.

7. Roots and branch cuts. There is one class of functions which we have already extended to all complex numbers but whose derivatives we have not yet discussed, namely the roots. It turns out that a study of these functions reveals a subtlety in functions of a complex variable which is not visible in functions of a real variable. Let us fix some positive integer m and consider mth roots. Recall that if $z = r(\cos\theta + i\sin\theta)$ is any complex number, then the m complex numbers

$$w_n = r^{1/m}\left(\cos\frac{\theta+2\pi n}{m} + i\sin\frac{\theta+2\pi n}{m}\right)$$

all satisfy $w_n^m = z$. Now a function must have a unique value at a given point; thus if we wish to define an mth root *function* we must have some way of choosing just one of these values for each point. At first sight it would appear that we could just take w_0 and be done, but a bit more thought reveals that the situation is not quite that simple: for example, should $z = r$, for r a positive real number, be represented as

$$z = r(\cos 0 + i\sin 0), \qquad \text{with } m\text{th root} \qquad w_0 = r^{1/m},$$

or as

$$z = r(\cos 2\pi + i\sin 2\pi), \qquad \text{with } m\text{th root} \qquad w_0 = r^{1/m}\left(\cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}\right)?$$

[2] For those who have seen this notation, we note that this is equivalent to saying that $\epsilon(h) = o(h)$ and $\epsilon'(h') = o(h')$.

If we are interested only in defining a function we may just choose one of these and be done. The problem with that method, though, is that the resulting function will not be continuous across the real axis. For suppose that we make the requirement that $\theta \in [0, 2\pi)$, which corresponds to choosing the first of these two expressions. Let us consider the two limits

$$\lim_{h\to 0^+}(\cos h + i\sin h)^{1/m} \qquad \text{and} \qquad \lim_{h\to 0^-}(\cos h + i\sin h)^{1/m}.$$

For our mth root function to be continuous these two limits must be equal. But since we have required the angle θ to lie in the interval $[0, 2\pi)$, we must rewrite the second number as

$$\cos(2\pi + h) + i\sin(2\pi + h)$$

(remember that here h is negative so $2\pi + h < 2\pi$!), which means that the two limits become

$$\begin{aligned}\lim_{h\to 0^+}(\cos h + i\sin h)^{1/m} &= \lim_{h\to 0^+}\left(\cos\frac{h}{m} + i\sin\frac{h}{m}\right)\\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lim_{h\to 0^-}(\cos(2\pi + h) + i\sin(2\pi + h))^{1/m} &= \lim_{h\to 0^-}\left(\cos\frac{2\pi + h}{m} + i\sin\frac{2\pi + h}{m}\right)\\ &= \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m},\end{aligned}$$

and these two expressions are clearly not equal unless $m = 1$ (when everything is quite trivial). A similar problem would happen if we made the second choice above.

It turns out that the above difficulty is not just a result of our lack of cleverness: there is in fact no way to define an mth root function which is single-valued and continuous on the entire complex plane. The basic idea is already contained in the foregoing. Suppose that $f : \mathbf{C} \to \mathbf{C}$ were a function of a complex variable satisfying everywhere on $\mathbf{C}$ the formula

$$[f(z)]^m = z,$$

and such that $f(z)$ were continuous everywhere on $\mathbf{C}$. Let us consider how f behaves on the unit circle. By our study of roots above, we know that there must be integer $n \in \{0, 1, 2, \cdots, m-1\}$ such that $f(1) = \cos\frac{2\pi n}{m} + i\sin\frac{2\pi n}{m}$. Since f is continuous, for θ close to zero we must also have

$$f(\cos\theta + i\sin\theta) = \cos\frac{\theta + 2\pi n}{m} + \sin\frac{\theta + 2\pi n}{m}.$$

Now let us consider what happens when we gradually increase θ more and more. Clearly we must always still have

$$f(\cos\theta + i\sin\theta) = \cos\frac{\theta + 2\pi n}{m} + \sin\frac{\theta + 2\pi n}{m},$$

since otherwise there would be a point where we would need to switch to a different value of n, and this would lead to a discontinuity in f (this could be shown analogously to how we argued above about discontinuity across the real axis). Thus we can keep on going up until we get close to 2π. But if θ is very close to 2π the above result gives

$$f(\cos\theta + i\sin\theta) = \cos\frac{\theta + 2\pi n}{m} + \sin\frac{\theta + 2\pi n}{m};$$

but since we can consider $\theta < 0$ as well as $\theta > 0$, we also have

$$f(\cos\theta + i\sin\theta) = f(\cos(\theta - 2\pi) + i\sin(\theta - 2\pi)) = \cos\frac{\theta + 2\pi(n-1)}{m} + \sin\frac{\theta + 2\pi(n-1)}{m},$$

a contradiction.

Let us sum up what we have shown: No matter which choice of mth root we choose, if we continue it along a curve which encloses the origin, it will come back as a different root when we come back to the original point. This phenomenon is actually quite common in the study of functions of a complex variable, and the origin is what is called a *branch point* of the mth root function. Far from being a failure of the theory, it actually leads to very interesting new mathematical structures called *Riemann surfaces*, which we discuss momentarily.

It turns out that if we wish to define an mth root function, there are two distinct ways to proceed. First of all, we could restrict the domain by removing (say) a ray from the origin to infinity from the domain of the function; for example, if we remove the positive real axis together with the origin, it is clear that we may make any single choice of n and get a continuous mth root function on the remaining set. The same is true if we remove any other ray from the origin to infinity. In this setting, the ray we remove from the domain of f is termed a *branch cut*. See Goursat, §6, especially the discussion around Figure 5; see also some additional discussion in §8 herein, below.

Goursat's discussion of cutting the plane relates to the notion of a Riemann surface, which is part of the second possible route out of our difficulties, namely extending the domain to a so-called *m-sheeted cover* of the complex plane.[3] This is rather complicated and we shall only sketch it. The idea is to consider the point 1 on the real axis as distinct from the point obtained by rotating it around the origin once, twice, thrice, $\ldots$, $m-1$ times, but as *the same* as what one gets by rotating m times.[4] This gives m different 'sheets' – in some sense, m different 'copies' of the complex plane – which are joined onto each other in some fashion (think of a spiral staircase which somehow ends up where it started); and we can then define the mth root function by choosing root n on the nth of the sheets.

[3] I don't suppose anyone has studied covering spaces, but in case anyone has, let me just note that this corresponds to the m-sheeted cover of the unit circle by itself. The universal cover of the unit circle will show up when we talk about the logarithm.

[4] If any of you have some familiarity with the notions of particle physics, you may recall that certain elementary particles, such as the electron, are said to have *spin-1/2*, in that they must be turned around *twice* to look the same (a most peculiar property!); that is exactly the same as what is going on here except that for mth roots we must 'turn around' m times to look the same. While it has been too long since I studied the Dirac equation to be sure of myself here, I doubt this is entirely a coincidence, as those of you who have studied the Dirac equation will probably recall that it arises as a *square-root* of the Klein-Gordon equation.

Summary:

- We clarify some matters related to branch cuts.
- We then fill in some points from the last set of lecture notes.
- Finally, we introduce power series and discuss how to extend the exponential and logarithm to complex numbers.

8. Roots and branch cuts, II. In the lecture notes from Tuesday, §7, we demonstrated that it is impossible to make a continuous choice of root on the entire complex plane, so that we either need to remove a part of the plane (make a *branch cut*) or embed the complex plane into a much larger set (the so-called *Riemann surface* of the function) in order to get a well-defined, continuous, single-valued function. In this section we will step back a bit to consider what all of this means, and why we are discussing it.

First of all, a philosophical point which will be useful to keep in mind at many other points in the course also. In mathematics there are some results or concepts which we study because they can be immediately used to solve problems, and there are other results or concepts which we study because they help deepen our understanding, even if they are not directly (or at least immediately) applicable to solving problems. In elementary calculus, for example, the product rule is of the first kind, as is the first derivative test; while the notion of a continuous function, or the extreme value theorem, are more of the second kind. In this class, methods for calculating residues, which we shall study later, are of the first kind; while branch cuts, which we are studying now, are of the second kind. We study them not so much because we need them immediately for applications, or because we can immediately solve problems about them, but because they help deepen our understanding of what an analytic function of a complex variable *is*, and how it might behave.[1]

With this in mind, let us go back and investigate exactly *why* we needed a branch cut in the first place. The most immediate answer is that we needed a branch cut to make sure we could keep our function continuous and single-valued. Why did it become multiple-valued in the first place?

Let z be any nonzero complex number, and suppose that $z = r(\cos\theta + i\sin\theta)$ is a polar form of z. Then clearly so is $r\left[\cos(\theta + 2\pi n) + \sin(\theta + 2\pi n)\right]$. Now consider the following diagram; the block on the left is to be read top to bottom, then left to right, and we use the abbreviation $\operatorname{cis}\theta$ for $\cos\theta + i\sin\theta$ (we will see very soon that $\operatorname{cis}\theta = e^{i\theta}$, of course):

$$\underbrace{\left.\begin{matrix} \cdots, & r\operatorname{cis}(\theta - 2\pi m), & r\operatorname{cis}\theta, & r\operatorname{cis}(\theta + 2\pi m), & \cdots \\ \cdots, & r\operatorname{cis}(\theta - 2\pi(m-1)), & r\operatorname{cis}(\theta + 2\pi), & r\operatorname{cis}(\theta + 2\pi(m+1)), & \cdots \\ \cdots, & r\operatorname{cis}(\theta - 2\pi(m-2)), & r\operatorname{cis}(\theta + 4\pi), & r\operatorname{cis}(\theta + 2\pi(m+2)), & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots, & r\operatorname{cis}(\theta - 2\pi), & r\operatorname{cis}(\theta + 2\pi(m-1)), & r\operatorname{cis}(\theta + 2\pi(2m-1)), & \cdots \end{matrix}\right\}}_{z} \overset{z\mapsto z^{1/m}}{\longrightarrow} \left\{\begin{matrix} r^{\frac{1}{m}}\operatorname{cis}\frac{\theta}{m} \\ r^{\frac{1}{m}}\operatorname{cis}\frac{\theta+2\pi}{m} \\ r^{\frac{1}{m}}\operatorname{cis}\frac{\theta+4\pi}{m} \\ \vdots \\ r^{\frac{1}{m}}\operatorname{cis}\frac{\theta+2\pi(m-1)}{m} \end{matrix}\right.$$

where each quantity on the left is equal to z, and where each line on the left maps under the mth root function to a single value on the right. The issue is that while each of the quantities on the left is a polar representation of the *same* complex number z, the m quantities on the right represent *distinct* complex numbers – namely, the m possible mth roots of z. This diagram indicates one way of looking at the issue: the mth root function is most naturally considered as acting on the polar representation of a complex number z, but it takes representations of the *same* complex number to representations of *distinct* complex numbers. The point of a branch cut is to allow us to single out a preferred choice of polar representation for z in such a way that the resulting mth root is uniquely defined. (In terms of the above diagram, such a choice corresponds to picking a specific row.)

For example, suppose that we take our branch cut along the positive real axis: then we may require the angle in any polar representation of z to lie in the interval $(0, 2\pi)$. Now suppose that we are given the complex number $z = -1$. Since the point corresponding to this number makes an angle of π radians with the positive real axis, we can write it as $z = \operatorname{cis}\pi$. Now we could equally well write $z = \operatorname{cis}(2k+1)\pi$ for

[1] I read recently somewhere – unfortunately I have forgotten where – that functions of a complex variable are essentially defined by their singularities. Of the three kinds of singularities we shall see in this course, namely poles, essential singularities, and branch points, branch points are the hardest to deal with; in other words, as far as singularities are concerned anyway, things get simpler from here on out!

any integer k; but our choice of interval $(0, 2\pi)$ for the angle requires us to use $z = \operatorname{cis}\pi$. The mth root we get in this case is then $z^{1/m} = \operatorname{cis}\pi/m$.

It is not hard to find other examples; we give two just to demonstrate the point. Suppose that we choose the same branch cut but now require the angle to lie in the interval $(2\pi, 4\pi)$; there is no reason why we can't do this. Then the point $z = -1$ will be represented as $z = \operatorname{cis} 3\pi$, and the corresponding choice of mth root will be $z^{1/m} = \operatorname{cis} 3\pi/m$.

Finally, suppose that we choose a different ray as our branch cut, say the positive *imaginary* axis. Our possible choices of intervals are different now: instead of avoiding the positive real axis, which has angle 0, we now need to avoid the positive imaginary axis, which has angle $\pi/2$. Thus we may choose an interval of the form $(-3\pi/2, \pi/2)$ (for example). In this case, the polar representation of z will be $z = \operatorname{cis}(-\pi)$, and the corresponding choice of mth root will be $z^{1/m} = \operatorname{cis}(-\pi/m)$.

To sum up: a branch cut determines the possible different choices of representation for z, and a selection of one of these makes the root function (or whatever other function we happen to be studying) single-valued.

Before moving on, I would like to emphasise again that the point of learning about branch cuts at this point is not because we are going to use them right away to solve problems (though we will see that they do come up in practical problems later on in the course), nor is it because we are going to immediately be able to go off and determine where functions have branch points. (Another, more involved, example of branch cuts is however given in the second part of §6 of Goursat.) Rather it is to be given an introduction to a particular feature of certain functions of a complex variable which we shall study more later.

– See §§5 and 6 above –

9. Conjugate harmonic functions [continuing §5]. Recall that we have shown in §5 above that the Cauchy-Riemann equations imply that the real and imaginary parts of an analytic function f must satisfy Laplace's equation[2] $\Delta u = 0$. However, the Cauchy-Riemann equations have more information than just this, as they give also a relationship between the real and imaginary parts. Thus we may consider the following question: Suppose that $P(x, y)$ is a real-valued function of two real variables which satisfies Laplace's equation; is there a function $Q(x, y)$ of two real variables such that

$$f(x + iy) = P(x, y) + iQ(x, y)$$

is an analytic function? It is not too hard to see that the answer is actually yes, at least if we stick to simply-connected regions. Let us write out the Cauchy-Riemann equations and see if we can solve them for Q:

$$\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}, \qquad \frac{\partial Q}{\partial y} = \frac{\partial P}{\partial x}. \tag{1}$$

Probably the most direct way to treat these equations is to use a bit of vector calculus. Let us define a vector field

$$\mathbf{F} = -\frac{\partial P}{\partial y}\mathbf{i} + \frac{\partial P}{\partial x}\mathbf{j};$$

then since P is harmonic we have

$$\operatorname{curl}\mathbf{F} = \frac{\partial}{\partial x}\frac{\partial P}{\partial x} - \frac{\partial}{\partial y}\left(-\frac{\partial P}{\partial y}\right) = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0,$$

which means that, as long as we stick to simply-connected regions (recall that these are regions 'without holes'; generally these are introduced when one studies Green's theorem), there must be a function $f(x, y)$ such that $\mathbf{F}(x, y) = \nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$. In other words, there must be a function f such that

$$\frac{\partial f}{\partial x} = -\frac{\partial P}{\partial y}, \qquad \frac{\partial f}{\partial y} = \frac{\partial P}{\partial x}.$$

[2] At least, assuming that they have continuous second-order partial derivatives. We shall see shortly that if a function f is analytic throughout a region – as opposed to at a single point – then this condition is always satisfied. As far as I know, functions which are analytic at isolated points are of interest only as mathematical curiousities, and have no particular use in applications, so we shall not generally worry about them.

But these are exactly the equations we wanted Q to satisfy; in other words, what we know from vector calculus shows us that there must be a solution Q to the equations (1). It is unique up to an additive constant.

To be more specific, recall that we also know from vector calculus that the function f can be written as

$$f(x, y) = \int_{(x_0,y_0)}^{(x,y)} \mathbf{F} \cdot d\mathbf{x} + C,$$

where (x_0, y_0) is any point in the domain of P, the integral is a line integral along any path joining the two points (it will not depend on this path because $\operatorname{curl} \mathbf{F} = 0$ implies that $\mathbf{F}$ is conservative) and C is any constant. (In vector calculus, of course, we take C to be a real constant. Here C can be any complex constant.) This allows us to write

$$Q(x, y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P}{\partial y}\, dx + \frac{\partial P}{\partial x}\, dy + C,$$

and finally

$$f(x + iy) = P(x, y) + i \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P}{\partial y}\, dx + \frac{\partial P}{\partial x}\, dy + C.$$

10. Power series. Let us recall a few facts about power series over the real numbers. A *power series* is an infinite series of the form

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k, \tag{2}$$

where $\{a_k\}$ is a sequence of coefficients, x_0 is some real number, and we consider x as a variable real number. The series will be *absolutely convergent* (meaning that the sum of the absolute values of its terms will be finite)[3] on some interval of the form $(x_0 - R, x_0 + R)$, called the *interval of convergence*, where $R > 0$ is called the *radius of convergence* and can be calculated from

$$\frac{1}{R} = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|,$$

when this limit exists.

Suppose now that we allow the numbers in the series in (2) to become complex. Now it turns out that, just as for real numbers, a series of complex numbers which is absolutely convergent is also convergent, so we may begin by asking where this series is absolutely convergent, which means that we must consider the series

$$\sum_{k=0}^{\infty} |a_k|\, |z - z_0|^k .$$

But this is just a power series of *real* numbers with coefficients $|a_k|$, and must therefore converge when $|z - z_0| < R$, where R is given as before. From this we can draw two conclusions:

1. Power series over the complex numbers converge in *discs*;
2. In the case that the coefficients a_k are all real, the radius of the disc of convergence is equal to the radius of the interval of convergence.

[3] The notion of absolute convergence is very important in more theoretical parts of analysis. Since a series of positive terms converges if and only if it has an upper bound, and since in most spaces in which these concepts make sense – and in particular, for real and complex numbers – an absolutely convergent series is convergent, we are to reduce a question of convergence of a series – which is hard – to the question of finding an upper bound for a series, which is generally simpler. We shall probably not have much need to use these concepts and results directly, however.

Point 2 in particular makes the term *radius* of convergence much more sensible!

Just as with real power series, power series of complex numbers can be added, multiplied (though that becomes messy very quickly, as anyone who has attempted such a procedure can surely attest!), and differentiated term-by-term. This means, *inter alia*, that power series represent analytic functions where they converge. Also as with real power series, a power series converges inside its disc of convergence and diverges outside; on the boundary, as with real power series, it may converge or diverge, depending on the point and the situation.[4] Our main interest with power series right now is that they provide a convenient way to extend the elementary transcendental functions (the exponential, trigonometric, and logarithmic functions) to complex numbers, which we take up now.

11. Exponentials and logarithms of a complex variable. Recall that the exponential function e^x has the power series representation

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k,$$

and that this series converges for all real numbers x. By our discussion above, this shows that the power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} z^k,$$

where z is now a *complex* variable, must converge for all complex numbers z. It is clearly equal to e^x when $z = x$ is a real number. Now it can be shown (and we shall probably be able to show this in the second half of the course) that analytic functions are incredibly *rigid*: roughly, if they are equal on any set which is not somehow 'discrete', they must be equal everywhere. (We shall make this more precise later as it is not exactly true as it stands.)[5] This suggests that the above power series of complex numbers, which as we have seen defines a function which is analytic everywhere on the complex plane, is the *unique* function analytic everywhere on the complex plane which is equal to the ordinary exponential function on the real axis. We thus define, for any complex number z, the complex exponential

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

When convenient for typographical reasons we may write $\exp z$ instead of e^z. The standard properties of exponential functions can be shown to follow from this expansion; for example, if z_1 and z_2 are any complex numbers, we have

$$\begin{aligned}
e^{z_1} e^{z_2} &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} z_1^k \right) \left(\sum_{\ell=0}^{\infty} \frac{1}{\ell!} z_2^\ell \right) \\
&= \sum_{k,\ell=0}^{\infty} \frac{1}{k!\ell!} z_1^k z_2^\ell \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} z_1^k z_2^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (z_1 + z_2)^n = e^{z_1+z_2},
\end{aligned}$$

[4] We note that it is possible to find a function which is analytic everywhere inside a disc but at no point of the boundary.

[5] For those who know enough topology to understand the following, we note that two analytic functions which agree on a set with at least one accumulation point must agree on the connected component of the intersection of their domains containing that set.

where in the third line we have introduced the variable $n = k + \ell$.

We know that on the real axis e^z agrees with the ordinary exponential function; what happens on the imaginary axis? Let $z = iy$; then we have

$$\begin{aligned} e^z = e^{iy} &= \sum_{k=0}^{\infty} \frac{1}{k!}(iy)^k \\ &= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!}(iy)^{2\ell} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}(iy)^{2m+1} \\ &= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!}(-1)^{\ell} y^{2\ell} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} i(-1)^m y^{2m+1} \\ &= \cos y + i \sin y, \end{aligned}$$

probably one of the most fascinating results in mathematics. This formula makes much of our work with powers and roots far more transparent: for example, the result

$$[r(\cos\theta + i\sin\theta)]^{1/m} = r^{1/m}(\cos\theta/m + i\sin\theta/m)$$

(where we have chosen just one particular mth root for simplicity) becomes now

$$\left[re^{i\theta}\right]^{1/m} = r^{1/m} e^{i\theta/m},$$

which is exactly what we would expect were the standard rules of exponents applicable to the complex exponential function.

Having now defined the exponential function for all complex numbers, we proceed to consider the logarithm. From what we have just seen, an arbitrary nonzero complex number z can be written in the form

$$z = re^{i\theta}$$

for some real number $r > 0$ ($r \neq 0$ since z is nonzero) and some real number θ. But since $r > 0$ we have

$$r = e^{\log r},$$

where here log represents the ordinary logarithm of positive real numbers; thus we can write

$$z = e^{\log r + i\theta}.$$

Now the defining property of the logarithm on real numbers is, that it is the inverse of the exponential function; if we wish to define the logarithm of a complex number the same way, the above formula suggests that we should define it to be $\log r + i\theta$. But here we run into the same problem we found when we discussed roots: θ is only defined up to an integer multiple of 2π. Thus for complex numbers we must evidently define the logarithm to be a multi-valued function. With this in mind, we define the logarithm of a nonzero complex number z, which we write $\operatorname{Log} z$, to be the collection of numbers

$$\operatorname{Log} z = \log r + i\theta,$$

where $r = |z|$ is the modulus of z and θ is any value of the argument of z. As with roots, this means that the logarithm has a branch point at the origin, and we must make a branch cut in order to get a single-valued continuous logarithm.

With these functions now defined, we may define exponents of any (nonzero) complex base and any complex power. First we recall that if $x_1 > 0$ and x_2 are two real numbers, we may write, by rules of exponents and logarithms (here log denotes the ordinary logarithm of positive real numbers)

$$e^{x_2 \log x_1} = e^{\log x_1^{x_2}} = x_1^{x_2}.$$

Now if we use the complex logarithm Log defined above, we can compute the left-hand side of the above equation for all *complex* numbers z_1 and z_2, as long as $z_1 \neq 0$. Thus, let $z_1 \neq 0$ and z_2 be two complex numbers; then we define

$$z_1^{z_2} = e^{z_2 \operatorname{Log} z_1}.$$

Note though that, since Log is multivalued, this definition in general makes $z_1^{z_2}$ a multivalued function as well. This leads to some rather amusing results. Let us give some examples.

EXAMPLES. 1. Before giving the amusing examples, let us first see how this definition fits in with the exponents we have already studied, namely integer powers and roots. If m is a positive integer and $z = r(\cos\theta + i\sin\theta)$ is any nonzero complex number, the above definition gives

$$z^m = e^{m \operatorname{Log} z} = \exp\left(m[\log r + i\theta]\right) = \exp\left(m\log r + im\theta\right) = e^{m\log r}e^{im\theta} = r^m(\cos m\theta + i\sin m\theta),$$

exactly in accord with our previous definition. Note that in this particular case the exponential function is *single*-valued, since if θ' is any other value of the argument of z, we would have $\theta' - \theta = 2\pi k$ for some integer k, and the above formula would give

$$r^m(\cos m\theta' + i\sin m\theta') = r^m(\cos m(2\pi k + \theta) + i\sin m(2\pi k + \theta)) = r^m(\cos m\theta + i\sin m\theta)$$

as before.

Let us now consider roots. Thus, again, let m be a positive integer and $z = r(\cos\theta + i\sin\theta)$ a nonzero complex number; then we have

$$z^{\frac{1}{m}} = \exp\left(\frac{1}{m}[\log r + i\theta]\right) = \exp\left(\frac{1}{m}\log r\right)\exp\left(i\frac{\theta}{m}\right) = r^{1/m}\left(\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\right),$$

exactly in accord with our original definition of mth roots. Recall that here θ represents *any* possible argument value for z, so that this expression represents all possible mth roots and is, as usual, multivalued for $m \neq 1$.

More generally, if $z' = \frac{k}{m}$ where k and m are relatively prime integers (meaning that they have no common divisors; this restriction is for convenience only), then we have for any complex number $z = re^{i\theta}$

$$z^{z'} = z^{k/m} = \exp\left(\frac{k}{m}[\log r + i\theta]\right) = \exp\left(\frac{k}{m}\log r + i\frac{k\theta}{m}\right) = r^{k/m}\left(\cos\frac{k\theta}{m} + \sin\frac{k\theta}{m}\right).$$

2. Now for some amusing examples. Let us recall that the exponential for real numbers is only defined for *positive* bases. We now have a means of defining it for arbitrary *complex* bases, but in particular for *negative* real bases; what does it give us? In particular, what is say -1 raised to an irrational power, say $\sqrt{2}$? To find this, we write $-1 = \cos(2n+1)\pi = e^{(2n+1)\pi i}$, where n is any integer; then we have

$$\begin{aligned} -1^{\sqrt{2}} &= \exp\left(\sqrt{2}\operatorname{Log}(-1)\right) = \exp\left(\sqrt{2}(2n+1)\pi i\right) \\ &= \cos\left(\sqrt{2}(2n+1)\pi\right) + i\sin\left(\sqrt{2}(2n+1)\pi\right). \end{aligned}$$

What does this set of numbers look like? It turns out that this set is actually *infinite*; this is because $\sqrt{2}$ is irrational: if the set were finite, we would have integers $n \neq m$ and k such that

$$\sqrt{2}(2n+1)\pi = \sqrt{2}(2m+1)\pi + 2k\pi,$$

which would give $\sqrt{2} = \frac{k}{n-m}$, contradicting irrationality of $\sqrt{2}$. It is also clear that all of these numbers lie on the unit circle; thus we have an infinite set of numbers on the unit circle, which means that they cannot be 'evenly spaced' in any meaningful sense. (For those who are familiar with the concept of density, we note that this set is in fact *dense* in the unit circle.)

Even more bizarre things happen when we look at *complex* bases. For example, let us consider i^i. Writing $i = \exp i\left(\frac{\pi}{2} + 2n\pi\right)$, we have

$$i^i = \exp\left(i\left[i\left(\frac{\pi}{2} + 2n\pi\right)\right]\right) = \exp\left(-\frac{\pi}{2} - 2n\pi\right),$$

i.e., the number i^i is an infinite sequence of *real* numbers!

(We hasten to note that these examples are more amusing than indicative, and while it is important to keep in mind that exponentials like $z_1^{z_2}$ can be very ill-behaved compared with their real counterparts, this behaviour will not generally concern us in the remainder of the course.)

Summary:

- We discuss the branches of the logarithm function defined previously and show how to differentiate them.
- We introduce the extension of the trigonometric functions to the complex plane, and relate them to the ordinary trigonometric and hyperbolic trigonometric functions of a real variable.
- We show how the inverse trigonometric functions can be determined in terms of roots and logarithms, and calculate their derivatives.
- Finally, we give a slightly more careful description of the kind of region we assume our functions are defined; then we give an introduction to *conformal mappings* and show that analytic functions are conformal.

12. Differentiation of Log. Recall that we have defined the complex logarithm as a multi-valued function as follows. If z is any nonzero complex number and $re^{i\theta}$ is any polar representation of z, then we define

$$\operatorname{Log} z = \log r + i(\theta + 2n\pi), \qquad n \in \mathbf{Z},$$

where here log denotes the ordinary real logarithm of a positive real number. (Note that this definition allows us to extend the logarithm to negative real numbers but *not* to zero. Since even over the complex plane the exponential is never 0, there is no way to extend the logarithm to zero.) As for the root functions we studied previously, a single-valued, continuous logarithm can only be defined on a cut plane. Let us see how this works in practice. Suppose that we cut the plane along the ray $\theta = \theta_0$, i.e., that we define the logarithm only on complex numbers with polar representation $z = re^{i\theta}$ where $\theta \in (\theta_0, \theta_0 + 2\pi)$, and that we consider only this polar representation in defining the logarithm. (Note that, while related, these are two distinct points.) Then we have

$$\operatorname{Log} z = \log r + i\theta.$$

We note that this function is continuous on the cut plane; an outline of a proof is given in the appendix. Some examples related to this are given in the problem set.

Let us now see whether these branches of Log are analytic functions. Specifically, let us take the above branch, obtained by cutting the plane along $\theta = \theta_0$. We shall denote this particular branch by $\operatorname{Log} z$ in the following, for convenience. We must determine whether the limit

$$\lim_{h\to 0} \frac{\operatorname{Log}(z+h) - \operatorname{Log}(z)}{h}$$

exists. This limit may clearly be written as

$$\lim_{z'\to z} \frac{\operatorname{Log} z' - \operatorname{Log} z}{z' - z}.$$

Now if $z = re^{i\theta}$, where $\theta \in (\theta_0, \theta_0 + 2\pi)$, then as long as z' is close enough to z[1] we may write $z' = r'e^{i\theta'}$ where $\theta' \in (\theta_0, \theta_0 + 2\pi)$ and also θ' is close to θ. Let us now define

$$w = \operatorname{Log} z = \log r + i\theta, \qquad w' = \operatorname{Log} z' = \log r' + i\theta'.$$

Then

$$\frac{\operatorname{Log} z' - \operatorname{Log} z}{z' - z} = \frac{w' - w}{e^{w'} - e^{w}}.$$

Now as $z' \to z$, we have clearly (by continuity of the logarithm) $\operatorname{Log} z' \to \operatorname{Log} z$, i.e., $w' \to w$; and in this limit the above fraction becomes

$$\lim_{w'\to w} \frac{w' - w}{e^{w'} - e^{w}} = \lim_{w'\to w} \frac{1}{\frac{e^{w'} - e^{w}}{w' - w}} = \frac{1}{\lim_{w'\to w} \frac{e^{w'} - e^{w}}{w' - w}} = \frac{1}{e^{w}},$$

[1] Specifically, we need the angle between them to be less than the smaller of $\theta - \theta_0$ and $\theta_0 + 2\pi - \theta$.

since the exponential function is analytic and is equal to its own derivative. But recall that

$$e^w = e^{\operatorname{Log} z} = z,$$

so that we have shown that

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}.$$

Note that this final result does not depend on the choice of branch cut; in other words, each branch of Log has the same derivative. This accords with what we know about derivatives from ordinary calculus, since the various branches of Log differ only by constants.

To sum up, we have shown that each branch of Log is an analytic function on its domain, and all of the branches have the same derivative, namely $1/z$.

Appendix I. Continuity of Log. Let us show that each branch of the logarithm, as outlined at the start of the section above, is in fact continuous. We shall give a formal ϵ-δ argument, but provide intuitive commentary to hopefully make the ideas clear to those who do not have much background in such things. Thus let $z = re^{i\theta}$ be an element of the cut plane, with $\theta \in (\theta_0, \theta_0 + 2\pi)$, and let $\epsilon > 0$. We may assume that $\epsilon < \frac{\pi}{4}$. Since log is continuous on the positive real line, there must be a $\delta' > 0$ such that

$$|\log r - \log r'| < \frac{1}{2}\epsilon \qquad \text{if} \qquad |r - r'| < \delta';$$

in other words, if r' is close to r then $\log r'$ is close to $\log r$. Further, it can be shown that the function $z \mapsto |z|$ is continuous; thus there is a $\delta'' > 0$ such that

$$||z| - |z'|| < \delta' \qquad \text{if} \qquad |z - z'| < \delta'';$$

in other words, $|z|$ is close to $|z'|$ if z is close to z' (clearly a reasonable statement geometrically!). Dealing with the angular part of z and z' is slightly messy; intuitively though the result is clear: if z' is sufficiently close to z, then we may write $z' = r'e^{i\theta'}$ where $\theta' \in (\theta_0, \theta_0 + 2\pi)$ and θ' is close to θ. To prove what we need carefully, though, let us set

$$\delta''' = \begin{cases} \frac{1}{2} r \sin(\theta - \theta_0), & \theta \in (\theta_0, \theta_0 + \pi/2) \cup (\theta_0 + 3\pi/2, \theta_0 + 2\pi), \\ \frac{1}{2} r, & \text{otherwise.} \end{cases}$$

Since $2\delta'''$ is simply the distance from z to the cut (draw a picture!), it is clear that $|z - z'| < \delta'''$ means that z' is on the same side of the cut as z, and hence can be written in the above form. Now let δ be the smaller of δ', δ'', δ''', and $\sin(\epsilon/2)$, and suppose that

$$|z - z'| < \delta.$$

By the foregoing, then,

$$||z| - |z'|| < \delta', \qquad \text{so} \qquad |\log|z| - \log|z'|| < \frac{1}{2}\epsilon;$$

furthermore, writing $z' = r'e^{i\theta'}$, $\theta' \in (\theta_0, \theta_0 + 2\pi)$, it is clear geometrically (again, draw a picture!) that the angle between z and z' is no greater than $\arcsin \delta$, which is bounded by $\epsilon/2$, so that $|\theta - \theta'| < \epsilon/2$. Thus finally

$$|\operatorname{Log} z - \operatorname{Log} z'| = |\log r + i\theta - \log r' + i\theta'| \leq |\log|z| - \log|z'|| + |\theta - \theta'| < \epsilon,$$

proving continuity of Log, as desired.

13. Trigonometric functions. To extend the trigonometric functions to the complex plane, we shall proceed in the same way we did with the exponential function. Recall that on the real line we have the power series expansions

$$\sin x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k x^{2k+1}, \qquad \cos x = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k x^{2k}.$$

Since the radius of convergence of both of these series is infinite, they must converge on the entire complex plane as well; thus we may define

$$\sin z = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k z^{2k+1}, \qquad \cos z = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k z^{2k},$$

where now z is any complex number. Moreover, as we mentioned in our discussion of the exponential function in section 11 above, these power series are the *unique* way of extending sin and cos to the complex plane as analytic functions.

The standard identities of trigonometry can be shown to hold over the complex numbers as well; in particular, we have

$$\cos^2 a + \sin^2 a = 1,$$
$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b, \qquad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,$$
$$\sin 2a = 2 \sin a \cos a, \qquad \cos 2a = \cos^2 a - \sin^2 a,$$

and so forth, where now a and b can be any complex numbers. Moreover, sin is odd ($\sin(-z) = -\sin z$) while cos is even ($\cos(-z) = \cos z$), as with real numbers. Further, the differentiation formulæ for sin and cos also hold. This can be shown by differentiating the above series:[2]

$$\frac{d}{dz}\sin z = \frac{d}{dz}\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k z^{2k+1} = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k z^{2k} = \cos z,$$
$$\frac{d}{dz}\cos z = \frac{d}{dz}\sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k z^{2k} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)!}(-1)^k z^{2k-1} = -\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k z^{2k+1} = -\sin z,$$

where we have set the lower index to 1 in the second series on the second line since the constant term in the series for $\cos z$ differentiates to zero, and we have adjusted the index in the last equality.

Now recall that, by substituting in to the power series expression for e^z, we found that when y is real

$$e^{iy} = \cos y + i \sin y.$$

Now there is nothing in this derivation which requires y to be a real number; thus with the above definitions for sin and cos, we find that for all *complex* numbers z that

$$e^{iz} = \cos z + i \sin z.$$

Using the fact that cos is odd and sin is even, we see that

$$e^{-iz} = \cos(-z) + i \sin(-z) = \cos z - i \sin z;$$

adding and subtracting these two equations, we obtain the results

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

This allows us to derive expressions for the real and imaginary parts of $\cos z$ and $\sin z$. First of all, note that if y is real (actually for all complex y if we define cosh and sinh in the usual way, but we are only interested in real y for the moment)

$$\cos iy = \frac{e^{-y} + e^{y}}{2} = \cosh y, \qquad \sin iy = \frac{e^{-y} - e^{y}}{2i} = i \sinh y,$$

[2] As noted previously, convergent power series can be differentiated term-by-term on their discs of convergence.

where as usual

$$\cosh y = \frac{e^y + e^{-y}}{2}, \qquad \sinh y = \frac{e^y + e^{-y}}{2}.$$

Thus if $z = x + iy$,

$$\begin{aligned}\cos z &= \cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y,\\ \sin z &= \sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y.\end{aligned}$$

Now since cosh and sinh are unbounded, this means in particular that cos and sin are unbounded along the imaginary direction. In particular, the inequalities $|\cos x| \leq 1$, $|\sin x| \leq 1$, which are true for real x, do *not* hold for complex numbers.

Similar results can be derived for the other trigonometric functions (tangent, cotangent, secant, and cosecant) but we shall not go into that here.

14. Inverse trigonometric functions. Let us see what we can find about the inverse trigonometric functions, given the foregoing. Let us first consider $\sin z$; or, since we are interested in finding its inverse, $\sin w$, where w is another complex variable. We have the relation

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

Now let us set $z = \sin w$ and see whether we can solve for w. We have

$$\begin{aligned}\frac{e^{iw} - e^{-iw}}{2i} &= z\\ e^{iw} - e^{-iw} &= 2iz\\ e^{2iw} - 1 &= 2ize^{iw}\\ e^{2iw} - 2ize^{iw} - 1 &= 0\\ e^{iw} &= \frac{1}{2}\left(2iz + \left(4(iz)^2 + 4\right)^{1/2}\right)\\ &= iz + \left(1 - z^2\right)^{1/2},\end{aligned}$$

where we have dispensed with the $\pm$ usually present in the quadratic formula since $\left(1 - z^2\right)^{1/2}$ is defined to mean both square roots. Thus we may write

$$w = \frac{1}{i}\mathrm{Log}\left[iz + \left(1 - z^2\right)^{1/2}\right].$$

In other words, whenever w is any of the (infinitely many) complex numbers indicated by the right-hand side of this equation, we must have $\sin w = z$. We thus define

$$\arcsin z = \frac{1}{i}\mathrm{Log}\left[iz + \left(1 - z^2\right)^{1/2}\right].$$

Note that there are, in general, *two* distinct sources of multi-valuedness in the above expression, one from the square root (when $z \neq \pm 1$) and the other from the log. This is in good accord with our understanding of the graph of $\sin x$ on the real line: as long as $y_0 \neq \pm 1$, the graph of $y = \sin x$ will intersect the line $y = y_0$ twice per interval of length 2π.

Similar expressions can be derived for arccos and arctan but we pass over them for the moment.

The above expression may be differentiated, assuming that we are using appropriate branches:

$$\begin{aligned}\frac{d}{dz}\frac{1}{i}\mathrm{Log}\left[iz + \left(1 - z^2\right)^{1/2}\right] &= \frac{1}{i}\frac{1}{iz + (1-z^2)^{1/2}}\left(i - \frac{z}{(1-z^2)^{1/2}}\right)\\ &= \frac{1}{iz + (1-z^2)^{1/2}}\frac{\left(1 - z^2\right)^{1/2} + iz}{(1-z^2)^{1/2}} = \frac{1}{(1-z^2)^{1/2}},\end{aligned}$$

in accord with what we know from real-variable calculus (except recall that here the square root means *both* square roots, i.e., it has a sign ambiguity).

15. Regions; conformal mappings. We have mentioned that we are principally interested in functions which are analytic in some *region*, rather than at a single point. We have however not defined what kind of region we are interested in. We are interested in the first place in functions which are analytic everywhere inside a so-called *simple closed curve*, i.e., a closed curve which does not intersect itself; such a region is *simply-connected* in the sense in which that word is typically used in discussions of Green's theorem, namely, it does not have any *holes*.[3] Later we shall also consider functions which are analytic on a set which has a finite number of holes, i.e., whose boundary is a finite number of simple closed curves, which moreover do not intersect each other. Whenever we speak of an analytic function, we are assuming that the function is analytic throughout a region of this form.

We shall now introduce so-called *conformal mappings*. It will turn out that all analytic functions on the complex plane are conformal mappings whenever they have nonzero derivative, but the definition of a conformal mapping does not require any use of complex numbers. A map

$$f : \mathbf{R}^2 \to \mathbf{R}^2$$

is said to be *conformal* at a point p when it preserves angles at that point; in other words, it $\gamma_1(t)$ and $\gamma_2(t)$ are any two curves which intersect at p, which for convenience and without loss of generality we may take to be $t = 0$ for both curves, then the angle between $\gamma_1(t)$ and $\gamma_2(t)$ at $t = 0$ is equal to the angle between $f(\gamma_1(t))$ and $f(\gamma_2(t))$ at $t = 0$, in both magnitude and sign (i.e., we measure it in the same direction, either clockwise or counterclockwise).[4] (See figures $9a$ and $9b$ in Goursat for an illustration.) Note that, in general, a map must be at least differentiable (in the sense of real functions on the plane!) for the angle of the image curves to make sense. Some examples immediately come to mind.

EXAMPLES. 1. Since translations and rotations of the plane preserve distances, they also preserve angles, and hence give conformal transformations.

2. So-called *isotropic scalings* of the plane, i.e., maps

$$(x, y) \mapsto (ax, ay),$$

where $a = 0$, are also conformal maps. This will follow from our general result below.

The main application we shall make of conformal mappings is to find solutions of Laplace's equation, which we shall take up probably in the second half of the course. The main example of conformal maps for us is given by the following result:

If f is analytic and $f'(z_0) \neq 0$, then f is conformal at z_0.

This may be shown as follows. (Here we first give the derivation given in the lecture, and supplement it to fill in a hole; we follow this with a slightly more concise demonstration.) For convenience we treat complex numbers as though they were their corresponding points in the plane. Let $\gamma_1(t)$ and $\gamma_2(t)$ be two smooth curves which satisfy $\gamma_1(0) = \gamma_2(0) = z_0$. Then they have tangent vectors there

$$\mathbf{T}_1 = \gamma_1'(0), \qquad \mathbf{T}_2 = \gamma_2'(0),$$

and hence make an angle θ which satisfies

$$\cos\theta = \frac{\mathbf{T}_1 \bullet \mathbf{T}_2}{|\mathbf{T}_1||\mathbf{T}_2|},$$

[3] For those who have seen something of general topology, the main point is that we are interested in functions which are analytic on some connected, simply-connected open set.

[4] For those of you who know something of modern differential geometry, the curves $\gamma_1(t)$ and $\gamma_2(t)$ here are being used as proxies for tangent vectors.

where $\bullet$ denotes the dot product. Now since f is analytic, it is in particular differentiable (in the real-variable sense) as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$, and thus the curves $f \circ \gamma_1$ and $f \circ \gamma_2$ are also smooth; moreover they have tangent vectors

$$\mathbf{S}_1 = f'(z_0) \cdot \gamma_1'(0), \qquad \mathbf{S}_2 = f'(z_0) \cdot \gamma_2'(0),$$

where we treat γ_1 and γ_2 as though they were complex-valued, and $\cdot$ denotes multiplication of complex numbers. (The foregoing is a simple extension of the chain rule.) Thus the angle between these image curves, say θ', satisfies

$$\cos\theta' = \frac{\mathbf{S}_1 \bullet \mathbf{S}_2}{|\mathbf{S}_1||\mathbf{S}_2|}.$$

Now recall (see the first example in §2, notes of May 5, above) that if z and w are any two complex numbers, then the dot product of the vectors corresponding to z and w is equal to $\operatorname{Re}\overline{z}w$. Thus we may compute as follows:

$$\begin{aligned}\mathbf{S}_1 \bullet \mathbf{S}_2 &= \operatorname{Re}\overline{f'(z_0)\mathbf{T}_1} f'(z_0)\mathbf{T}_2 = \operatorname{Re}\overline{f'(z_0)} f'(z_0)\overline{\mathbf{T}_1}\mathbf{T}_2 \\ &= |f'(z_0)|^2 \operatorname{Re}\overline{\mathbf{T}_1}\mathbf{T}_2 = |f'(z_0)|^2 \,\mathbf{T}_1 \cdot \mathbf{T}_2.\end{aligned}$$

Since $|\mathbf{S}_1|$ can be computed in terms of a dot product, we see that

$$\begin{aligned}\cos\theta' &= \frac{\mathbf{S}_1 \bullet \mathbf{S}_2}{|\mathbf{S}_1||\mathbf{S}_2|} = \frac{|f'(z_0)|^2 \,\mathbf{T}_1 \cdot \mathbf{T}_2}{|f'(z_0)|\,|\mathbf{T}_1|\,|f'(z_0)|\,|\mathbf{T}_2|} \\ &= \frac{\mathbf{T}_1 \cdot \mathbf{T}_2}{|\mathbf{T}_1||\mathbf{T}_2|} = \cos\theta.\end{aligned}$$

This shows that θ and θ' have the same cosine. However this of course does not mean that they are equal. (This point was not mentioned in the lecture.) To show that they are actually equal, we recall also that if z and w are any two complex numbers, the *cross product* (more carefully, the $\mathbf{k}$ component of the cross product) of z and w is equal to $\operatorname{Im}\overline{z}w$. Now recall from vector calculus that the cross product in this case is also given by $|z||w|\sin\phi$, where ϕ is the angle between the vectors corresponding to z and w. The foregoing calculation shows, replacing Re by Im everywhere, that we must have $\sin\theta = \sin\theta'$. Since two angles which have the same sine and cosine must be equal up to some integer multiple of 2π, and this means for our purposes that they are the same angle, this shows that f must be conformal at z_0, as claimed.

A slightly more concise demonstration may be given as follows. (Those of you who are familiar with derivatives considered as linear maps can skip straight to the appendix where an even more concise proof is given.) Let $t > 0$ be small. Then the tangent vectors to γ_1 and γ_2 at $t = 0$, i.e., at z_0, can be approximated by

$$\frac{\gamma_1(t) - z_0}{t}, \qquad \frac{\gamma_2(t) - z_0}{t}.$$

Similarly, the tangent vectors to $f(\gamma_1(t))$ and $f(\gamma_2(t))$ can be approximated by

$$\frac{f(\gamma_1(t)) - f(z_0)}{t}, \qquad \frac{f(\gamma_2(t)) - f(z_0)}{t}.$$

Now for z near z_0 we may write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0),$$

where $o(z - z_0)$ denotes a quantity which vanishes faster than $z - z_0$ as the latter goes to zero; i.e.,

$$\lim_{z \to z_0} \frac{o(z - z_0)}{z - z_0} = 0.$$

Thus we have

$$f(\gamma_k(t)) = f(z_0) + f'(z_0)(\gamma_k(t) - z_0) + o(\gamma_k(t) - z_0),$$

so

$$\frac{f(\gamma_k(t)) - f(z_0)}{t} = f'(z_0)\frac{\gamma_k(t) - z_0}{t} + \frac{o(\gamma_k(t) - z_0)}{t}.$$

Now in the limit $t \to 0$ we have similarly $\gamma_k(t) = \gamma_k(0) + \gamma_k'(0)t + o(t) = z_0 + \gamma_k'(0)t + o(t)$, so that in this limit the last quantity on the right-hand side above vanishes and we find that the tangent vector to the curves $\gamma_k(t)$ are given by

$$f'(z_0)\gamma_k'(0),$$

where as before the multiplication is to be considered as multiplication of complex numbers. Now suppose that we have

$$\gamma_k'(0) = r_k e^{i\theta_k},$$

and that

$$f'(z_0) = re^{i\theta};$$

then the tangent vectors to the image curves are given by

$$f'(z_0)\gamma_k'(0) = rr_k e^{i(\theta_k+\theta)};$$

in other words, the effect of an analytic map f on tangent vectors to smooth curves is to scale and rotate, which clearly preserves angles. This shows that f is conformal at z_0, as claimed.

Appendix I. Abstract derivation. Let us consider f as a map of the real plane. Then its derivative $f'(z_0)$ is a linear map from the plane to itself which satisfies

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|),$$

where here $f'(z_0)$ is considered as a linear map and $z - z_0$ as a vector, and the 'product' above is the application of this linear map to this vector. Evidently, $f'(z_0)$ may be considered to be multiplication by the complex derivative also denoted $f'(z_0)$. Now abstractly the derivative as a linear map takes tangent vectors to tangent vectors; in other words, two tangent vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ (say) at the point z_0 are taken by the map f to the vectors $f'(z_0)\mathbf{T}_1$ and $f'(z_0)\mathbf{T}_2$. By the discussion in the last few lines of the section above, the angle between these vectors must be that between $\mathbf{T}_1$ and $\mathbf{T}_2$.

(I admit that this is a little bit hand-wavy. The reason for this is that the definition of 'conformal' given above is somewhat informal. The argument just given can be made entirely rigorous if we define 'preserves angles at a point' to mean that its derivative preserves angles as a map of tangent vectors, which is more or less equivalent to the definition in terms of curves given above.)

Summary:

- We continue to discuss conformal mappings and expand on a couple examples from Goursat.

16. Examples of conformal maps. (a) (See Goursat, §19, Example 2.) Consider the map on the punctured plane $\mathbf{R}^2\backslash\{(0,0)\}$ which is given in complex notation by

$$f(z) = \frac{1}{z}.$$

Since this function is analytic on the punctured plane, it must be conformal at every point other than the origin. Let us consider how it behaves with respect to the unit circle. We have the following properties:

$$\begin{aligned} \text{If } |z| = 1 \text{ then } |f(z)| &= \left|\frac{1}{z}\right| = \frac{1}{|z|} = 1. \\ \text{If } |z| > 1 \text{ then } |f(z)| &= \left|\frac{1}{z}\right| = \frac{1}{|z|} < 1. \\ \text{If } |z| < 1 \text{ then } |f(z)| &= \left|\frac{1}{z}\right| = \frac{1}{|z|} > 1. \end{aligned}$$

This means that the map f takes the unit circle to itself, while it takes the region *outside* the unit circle to the region *inside* the unit circle, and vice versa. See Fig. 1. It is worth noting that on the unit circle

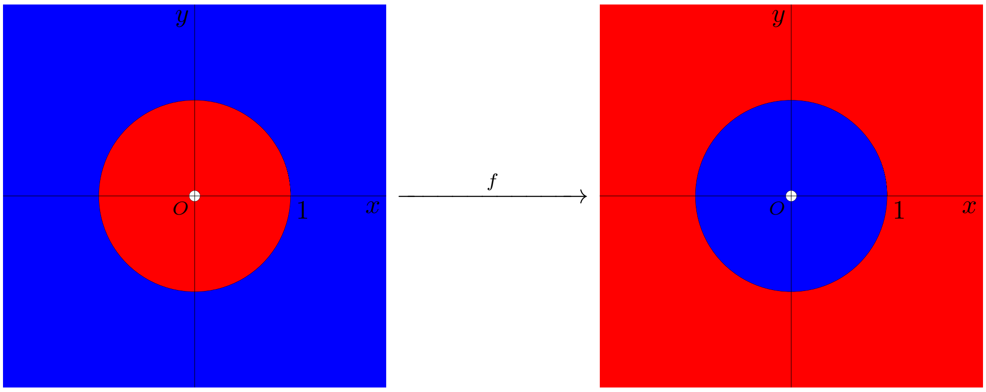

FIG. 1

$$f(z) = \frac{1}{z} = \overline{z}.$$

Note though that $\overline{z}$ is *not* an analytic function in general! It does turn out to be (almost) conformal though (it preserves magnitudes of angles but reverses their sense); and it can be shown (see §21 of Goursat, noting that replacing Q by $-Q$ is equivalent to taking the complex conjugate of f) that every sufficiently smooth conformal map is either an analytic function or the conjugate of an analytic function (which is the same thing as an analytic function of $\overline{z}$, as is apparent if one thinks of a Taylor expansion: but that is a bit beyond what we have technically covered so far).

(b) (See Goursat, §22, Example 2.) Let us now consider the function on the entire plane given in complex notation by

$$f(x+iy) = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y.$$

This function is analytic everywhere, and will be conformal everywhere that its derivative is nonzero. (We pause for a moment to clarify a point which the author fumbled during lecture. The derivative of $\cos z$ is $-\sin z$, which means that $\cos z$ will be conformal at every point where $\sin z$ is nonzero. Now

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y;$$

and for this to be zero, we see first of all that we must have $\sin x = 0$ (since $\cosh y \geq 1$ for all real y), and since this means that $\cos x \neq 0$, we must have $\sinh y = 0$, or $y = 0$. Thus the zeros of $\sin z$ over the complex plane are the same as those of $\sin x$ over the real line, i.e., $n\pi$, $n \in \mathbf{Z}$.) Thus f will be conformal at every point inside the strip $\{x+iy|0<x<\pi, y>0\}$. Let us consider how f maps straight lines within this strip. Let us consider first a horizontal line, say $y = y_0 > 0$. On such a line, f is equal to

$$f(x+iy_0) = \cos x \cosh y_0 - i \sin x \sinh y_0,$$

where $x \in (0, \pi)$. Now this is just another way of writing the parametric curve

$$t \mapsto (\cosh y_0 \cos t, -\sinh y_0 \sin t), \qquad t \in (0, \pi).$$

If we denote this curve by $(x(t), y(t))$ (where unfortunately here $x(t)$ and $y(t)$ are completely distinct from the real and imaginary parts of z), then we have

$$\left(\frac{x(t)}{\cosh y_0}\right)^2 + \left(\frac{y(t)}{\sinh y_0}\right)^2 = 1,$$

i.e., the curve must lie on an ellipse with major axis $\cosh y_0$ along the horizontal axis and minor axis $\sinh y_0$ along the vertical axis, and centred at the origin. Now since $y_0 > 0$, $\sinh y_0 > 0$, so $-\sinh y_0 < 0$ and $y(t) < 0$ for all $t \in (0, \pi)$, while $x(t)$ takes on all values from $\cosh y_0$ to $-\cosh y_0$. Thus we obtain the lower half of this ellipse.

Now let us consider a vertical line, say $x = x_0 \in (0, \pi)$. Working as before, we see that on this line

$$f(x_0 + iy) = \cos x_0 \cosh y - i \sin x_0 \sinh y.$$

If $x_0 = \pi/2$ then $\cos x_0 = 0$ and this is simply a parametrisation of the negative imaginary axis. Otherwise, we again write

$$(x(t), y(t)) = (\cos x_0 \cosh t, -i \sin x_0 \sinh t), \qquad t \in (0, \pi)$$

and note that (this follows from the basic identity $\cosh^2 x - \sinh^2 x = 1$)

$$\left(\frac{x(t)}{\cos x_0}\right)^2 - \left(\frac{y(t)}{\sin x_0}\right)^2 = 1,$$

which means that the curve lies on a hyperbola opening along the real axis with intercept $\pm \cos x_0$ and with asymptotes having slope $\pm \tan x_0$. Now we note that $y(t) < 0$ for all t, while $x(t) > 0$ for $t \in (0, \pi/2)$ and $x(t) < 0$ for $t \in (\pi/2, \pi)$; thus in the first case we have the lower right-hand portion of the hyperbola, while in the second case we have the lower left-hand portion. See Fig. 2. Note especially how the blue and red

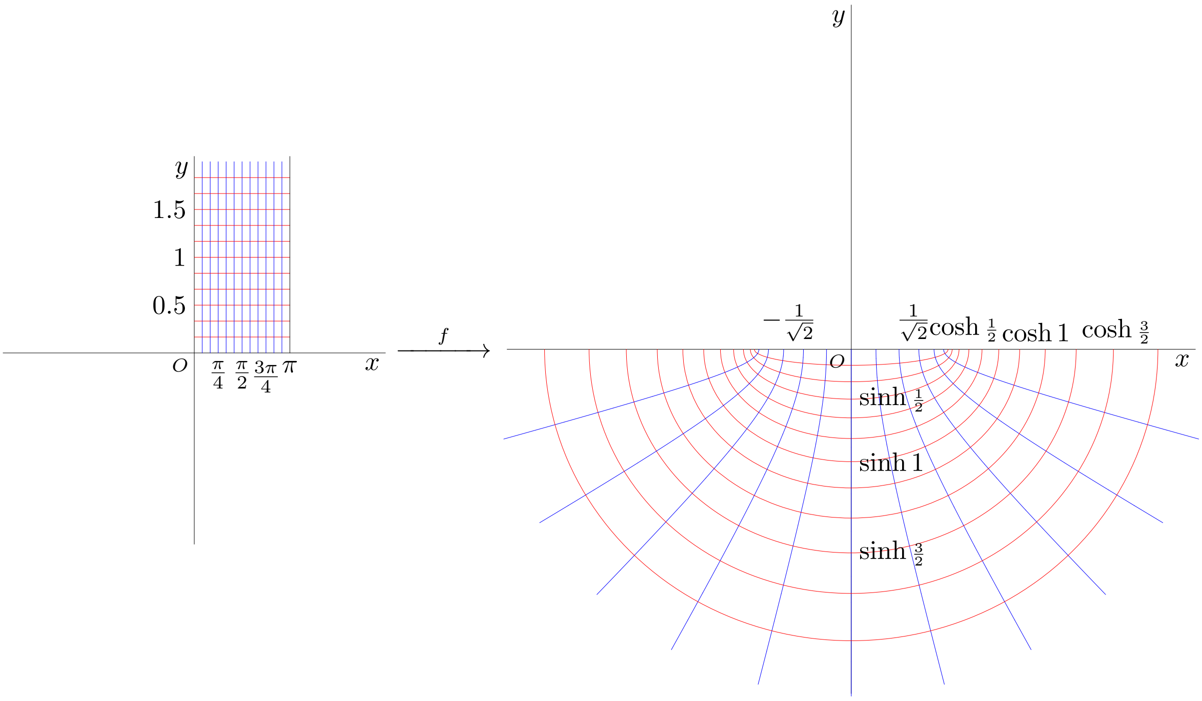

FIG. 2

curves on the right intersect at right angles, exactly like those on the left.

Summary:
- We outline a particular application of conformal maps.
- We then define and investigate integrals of complex functions over curves in the complex plane.

(Goursat, §§24, 25 – 26, 32.)

17. Application of conformal maps to harmonic functions. In fields as varied as electrostatics, heat flow, and fluid mechanics (and probably others) one is often interested in solving problems of the following form: we are given a particular region U in the plane[1] with boundary curve C, and some particular function g on the boundary curve C, and we wish to find a function P on U which satisfies

$$\Delta P = 0 \,\text{on}\, U, \qquad P|_C = g.$$

This problem in full generality is a topic for a course in partial differential equations, but there are specific cases which can be treated by using complex variable techniques to replace the region U by another one for which the problem is more tractable. Let us see how this works. (Here we shall simply outline the idea; we shall go over it in more detail later on in the course. Thus what follows is meant to be more of a conceptual introduction than a careful exposition.) Suppose that we have another region U' with boundary curve C' and a conformal map $f : U' \to U$ which maps C' onto C and is also analytic with an analytic inverse $f^{-1} : U \to U'$ (I accidentally forgot about this restriction in the lecture; there are probably ways of treating the problem without it, but we shall leave a detailed discussion of the matter for another time). Then we may consider instead the problem

$$\Delta P' = 0 \,\text{on}\, U', \qquad P'|_{C'} = g \circ f.$$

For a suitable choice of U' and f, this problem may be easier to solve than the original one. Suppose that we are able to find a solution to this modified problem. Then we claim that $P' \circ f^{-1}$ is a solution to the original problem. To see this, let $z_0 = x_0 + iy_0 \in U'$ be some particular point and define

$$Q' = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P'}{\partial y}\, dx + \frac{\partial P'}{\partial x}\, dy,$$

i.e., the conjugate harmonic function to P'; then the function

$$F(x + iy) = P'(x, y) + iQ'(x, y)$$

will be analytic, at least on any simply connected subset of U' containing (x_0, y_0). Since f^{-1} is also analytic, this means that $F \circ f^{-1}$ is also analytic, and hence that its real part

$$P' \circ f^{-1}$$

is harmonic.[2] Further, on C

$$\left(P' \circ f^{-1}\right)\big|_C = P'|_{C'} \circ f^{-1} = g \circ f \circ f^{-1} = g,$$

so that the boundary condition is satisfied as well. Thus $P' \circ f^{-1}$ is indeed a solution to the original problem, as claimed.

(In the lecture I actually showed the *opposite* implication, namely that if P is a solution to the original problem, then $P \circ f$ is a solution to the modified problem. The argument is identical to that here, replacing f^{-1} by f and P' by P as appropriate.)

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[1] While not strictly necessary, we can assume that U is simply connected below to avoid some technical complications which are not important at the moment.

[2] Note that we can afford to be vague about the region here since the property of being analytic and – especially – harmonic is really a pointwise property; or if we want to be a bit more careful, it is a property we only need to consider on small disks, which are always simply connected.

Appendix I. Formal definition of simple connectedness. Informally, a region which is simply connected is one which has no 'holes'. As mentioned in lecture, if the region is given to us pictorially this is about all we could go on to determine whether it is simply connected. More carefully, though, a region is simply connected if any closed curve can be continuously shrunk to a point within the region. But what do we mean by 'continuously shrunk to a point within the region'?

The precise definition of simply connected, which is valid in any topological space, is as follows. A set U is said to be *simply connected* if for any continuous closed curve $\gamma : [0,1] \to U$ (i.e., γ is continuous and satisfies $\gamma(0) = \gamma(1)$) there is a continuous map $F : [0,1] \times [0,1] \to U$ satisfying the following conditions:

$$\begin{aligned}
&F(\cdot, s) : [0,1] \to U \text{ is a closed curve in } U \text{ for each } s \in [0,1] \\
&F(t, 0) = \gamma(t) \text{ for all } t \in [0,1] \\
&F(t, 1) = u_0 \text{ for all } t \in [0,1]
\end{aligned}$$

where $F(\cdot, s)$ means the map $t \mapsto F(t,s)$, and $u_0 \in U$ is some point. If we unwrap this definition a bit, what it means is that $F(t,s)$ is a family of continuous, closed curves, where the curves are paremeterised by t and the family by s, such that the first curve in the family (when $s = 0$) is the original curve γ and the final 'curve' in the family (when $s = 1$) is a single point. More informally, F shows us specifically how to continuously deform γ to a single point within the region.

(It is probably worth pointing out here that the definition of simply connected as meaning 'without holes' only works in two dimensions. If we consider a ball, for example, and remove a single point, the resulting set clearly has a whole, but it is also clearly possible to shrink any continuous curve to a point regardless of the hole. (Think about it for a bit if it isn't clear!) These 'higher-dimensional' holes lead to so-called 'higher homotopy groups', or, more tractably, to homology theory – which I think actually originated in the study of functions of a complex variable!)

18. Complex integration. We now enter into one of the core parts of the course, the notion of *contour integrals* in the complex plane. We shall introduce these in the same way as done in Goursat (§25) and then show how they may be computed by reducing to the line integrals one studies in multivariable calculus.

Recall that in one-variable calculus we define the definite integral of a function f between points a and b more or less as follows:

$$\int_a^b f(x)\,dx = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k,$$

where $\Delta x_k = x_k - x_{k-1}$ and $x_k^* \in [x_{k-1}, x_k]$.[3] Now when we defined derivatives of functions on the complex plane, we were able to proceed by using essentially the same definition we used in the case of real-variable calculus. Let us see whether the same thing can be done in this case. Thus, let a and b be two *complex* numbers, and consider how we might adapt the limit definition above to this case. First of all we need to determine what is meant by the intermediate points between a and b; evidently we need a set of values z_1, z_2, $\cdots$, z_{n-1}. Now in the real case there is no real point in doing anything except going directly from the initial point to the final point; but in the complex plane there are many different paths which lead from a to b, and from what we have seen so far it is possible that these different paths may lead somehow to different results. Thus we evidently need to pick a path. Suppose that $\gamma(t)$ is a smooth (continuous with continuous derivative) path from a to b, and let $z_0 = a$, z_1, z_2, $\cdots$, z_{n-1}, $z_n = b$ be points along the curve ordered

[3] More precisely, for those of you who know something of ϵ–δ definitions, the limit above can be defined as follows: we say it is equal to L if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every *partition* $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$, $x_0 = a < x_1 < \cdots < x_n = b$ satisfying $\max\{|\Delta x_k| \,|\, k = 1, 2, \cdots, n\} < \delta$ and any set $\{x_k^* | k = 1, \cdots, n\}$ satisfying $x_k^* \in [x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(x_k^*) \Delta x_k - L \right| < \epsilon.$$

(Here, of course, $\Delta x_k = x_k - x_{k-1}$.)

by increasing parameter value. (Cf. Figure 12 in §25 of Goursat.) Then we define $\Delta z_k = z_k - z_{k-1}$, and consider the sum

$$\sum_{k=1}^{n} f(z_k^*)\Delta z_k$$

(where z_k^* is some point along the curve between z_{k-1} and z_k in parameter value). Now suppose that

$$f(x+iy) = P(x,y) + iQ(x,y)$$

and write

$$z_k^* = x_k^* + iy_k^*, \qquad \Delta z_k = \Delta x_k + i\Delta y_k;$$

then working out the above product, we have

$$\begin{aligned}\sum_{k=1}^{n} f(z_k^*)\Delta z_k &= \sum_{k=1}^{n}[P(x_k^*,y_k^*) + iQ(x_k^*,y_k^*)][\Delta x_k + i\Delta y_k] \\ &= \sum_{k=1}^{n} P(x_k^*,y_k^*)\Delta x_k - Q(x_k^*,y_k^*)\Delta y_k + i[P(x_k^*,y_k^*)\Delta y_k + Q(x_k^*,y_k^*)\Delta x_k] \\ &= \sum_{k=1}^{n} P(x_k^*,y_k^*)\Delta x_k - Q(x_k^*,y_k^*)\Delta y_k + i\sum_{k=1}^{n} P(x_k^*,y_k^*)\Delta y_k + Q(x_k^*,y_k^*)\Delta x_k.\end{aligned}$$

Now we wish to consider the limit of the above sum as $\Delta z_k \to 0$ (in the same sense as elaborated in the footnote above). Since $\Delta z_k = \Delta x_k + i\Delta y_k$, this is the same as the limit as Δx_k and Δy_k go to zero independently. Thus we may write

$$\begin{aligned}\lim_{\Delta z_k\to 0}\sum_{k=1}^{n} f(z_k^*)\Delta z_k = \lim_{(\Delta x_k,\Delta y_k)\to(0,0)}\sum_{k=1}^{n} &P(x_k^*,y_k^*)\Delta x_k - Q(x_k^*,y_k^*)\Delta y_k \\ &+ i\sum_{k=1}^{n} P(x_k^*,y_k^*)\Delta y_k + Q(x_k^*,y_k^*)\Delta x_k\end{aligned}$$

,

and we recognise these limits as giving ordinary line integrals of the form we have studied previously. In particular, the above limit becomes

$$\int_\gamma P(x,y)\,dx - \int_\gamma Q(x,y)\,dy + i\left[\int_\gamma P(x,y)\,dy + \int_\gamma Q(x,y)\,dx\right];$$

assuming, of course, that all of these integrals exist (as they will if P and Q are both continuous, for example). We take this as our definition and write

$$\int_\gamma f(z)\,dz = \int_\gamma P(x,y)\,dx - \int_\gamma Q(x,y)\,dy + i\left[\int_\gamma P(x,y)\,dy + \int_\gamma Q(x,y)\,dx\right].$$

Now suppose that γ is parameterised as $\gamma(t) = (x(t), y(t))$, $t \in [t_0, t_1]$; then the above can be written

$$\int_{t_0}^{t_1} P(x(t),y(t))x'(t) - Q(x(t),y(t))y'(t) + i[P(x(t),y(t))y'(t) + Q(x(t),y(t))x'(t)]\,dt.$$

Note that this is exactly what we would obtain if we were to replace dz in the integral $\int_\gamma f(z)\,dz$ with $x'(t)dt + iy'(t)dt$ and integrate from t_0 to t_1, i.e., if we were to pretend that the complex integral were simply another line integral with element $x'(t)dt + iy'(t)dt$. While this is not in itself a proof of anything, of course, it is useful for remembering the above formula; and it also suggests another mode of calculation: suppose that the curve γ is written in *complex* form as $z(t) = x(t) + iy(t)$; then the integral $\int_\gamma f(z)\,dz$ is equal to

$$\int_{t_0}^{t_1} f(x(t) + iy(t))(x'(t) + iy'(t))\,dt = \int_{t_0}^{t_1} f(z(t))z'(t)\,dt,$$

where we are useful to use *complex* techniques to determine $z'(t)$ and $f(z(t))$. In other words, this formula does not require us to split f into its real and imaginary parts, which is not convenient in many cases (such as when f is most usefully represented in terms of polar coordinates, for example).

The integral $\int_\gamma f(z)\,dz$ is called a *contour integral*.

19. First glimpse of the Cauchy integral theorem. Let us consider what happens when we integrate an analytic function over a closed curve. More specifically, suppose that we have a function $f(x + iy) = P(x, y) + iQ(x, y)$ which is analytic over a simply connected region U which has boundary curve C, and assume that C is oriented with respect to U as required by Green's theorem. Let us assume furthermore that the real and imaginary parts of f, namely P and Q, have continuous first-order partial derivatives. Then by applying Green's theorem and the Cauchy-Riemann equations, we have

$$\begin{aligned}\int_C f(z)\,dz &= \int_C P(x, y)\,dx - Q(x, y)\,dy + i\int_C Q(x, y)\,dx + P(x, y)\,dy \\ &= \int_U \frac{\partial}{\partial x}\,[-Q(x, y)] - \frac{\partial}{\partial y}\,[P(x, y)]\;dA + \int_U \frac{\partial}{\partial x}\,[P(x, y)] - \frac{\partial}{\partial y}\,[Q(x, y)]\;dA \\ &= \int_U -\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\,dA + \int_U \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\,dA = 0,\end{aligned}$$

since the Cauchy-Riemann equations give

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}.$$

Thus under the assumptions above, the contour integral of an analytic function over a closed curve is always 0. This is a central result in complex variable theory.

Unfortunately, the above demonstration, since it requires that the partial derivatives of the real and imaginary parts of f be continuous, is not sufficient for our purposes, since we actually want to use this result to *prove* continuity of those derivatives. Thus we shall soon see another proof of this result from first principles which does not make use of this assumption.

Summary:
- We fill in some holes in the previous exposition.
- We then proceed to give a proof of the Cauchy integral theorem which does not require continuity of the partial derivatives of the real and imaginary parts of the function.
- We show that analytic functions have antiderivatives, at least on simply-connected regions, which are also analytic, and discuss a connection with branch cuts.
- Finally, we discuss an extension of the Cauchy integral theorem to regions which are not simply connected.

(Goursat, §§28 – 31)

20. A few points from previous material. Recall that we have shown that, if m is a positive integer, then the *power rule* for differentiation on the real line applies also to derivatives in the complex plane:

$$\frac{d}{dz}z^m = mz^{m-1}.$$

The same result holds true for *any* complex exponent m, as long as we interpret the left- and right-hand sides appropriately. To see this, recall that if m is any complex number, we define the exponential z^m by

$$z^m = e^{m\,\mathrm{Log}\, z},$$

where $\mathrm{Log}\, z$ represents the full multivalued complex logarithm of the complex number z. As we discussed when we first gave this definition, the right-hand side is multivalued since Log is. Suppose now that we take a particular branch of Log, say by requiring the angle to lie between $(\theta_0, \theta_0 + 2\pi)$ for some $\theta_0 \in \mathbf{R}$.[1] For this particular branch, as in general,

$$\frac{d}{dz}\mathrm{Log}\, z = \frac{1}{z},$$

and by the chain rule we have

$$\frac{d}{dz}z^m = \frac{d}{dz}e^{m\mathrm{Log}\, z} = e^{m\mathrm{Log}\, z}\frac{d}{dz}m\mathrm{Log}\, z = me^{m\mathrm{Log}\, z}\frac{1}{z} = me^{m\mathrm{Log}\, z - \mathrm{Log}\, z} = me^{(m-1)\mathrm{Log}\, z} = mz^{m-1},$$

where z^{m-1} is taken using the same branch of Log as z^m.[2] Thus we do indeed have

$$\frac{d}{dz}z^m = mz^{m-1},$$

as long as the powers on both sides are computed using the same branch of the logarithm.

Since the functions $z \mapsto z^m$, where m is any nonzero integer, are all single-valued, the result above holds without any conditions for (nonzero) integer exponents. It holds for $m = 0$ if we define z^0 to be 1 everywhere, including at 0. (Recall that 0^0 is not defined.)

We now wish to point out another version of the chain rule involving complex numbers. Recall that, if f and g are two complex-valued functions of a complex variable, both of which are analytic, then $f \circ g$ is also analytic where it is defined, and we have

$$\frac{d}{dz}(f \circ g) = f'(g(z))g'(z).$$

Now suppose that f is an analytic function of a complex variable, and that $\gamma : [a, b] \to \mathbf{C}$ is a smooth curve. Then we have also

$$\frac{d}{dt}(f \circ \gamma) = f'(\gamma(t))\gamma'(t).$$

[1] It is worth noting here that, although we specify a branch cut for Log and for the root functions by specifying an interval for the angle θ, a branch cut is a cut of the *entire plane*, not just the unit circle.

[2] Note that it is *always* valid to write $z = e^{\mathrm{Log}\, z}$ and $\frac{1}{z} = e^{-\mathrm{Log}\, z}$, regardless of the branch of Log we are using (or even if we are not taking a branch at all). The first follows from the definition of Log as the inverse function of exp, and the second follows from the first by laws of exponents.

This can be shown in the same way that we showed the original chain rule above; briefly, we may write

$$f(\gamma(t+h)) = f(\gamma(t) + \gamma'(t)h + o(h)) = f(\gamma(t)) + f'(\gamma(t))(\gamma'(t)h + o(h)) + o(\gamma'(t)h + o(h)),$$

so if we are willing to accept that $o(\gamma'(t)h + o(h))$ is also $o(h)$, this becomes

$$f(\gamma(t+h)) = f(\gamma(t)) + f'(\gamma(t))\gamma'(t)h + o(h),$$

from which the result follows by computing the difference quotient and taking a limit. (Here, again, by $o(h)$ we mean any function – of a real variable in this case – which satisfies $\lim_{h\to 0} o(h)/h = 0$.)

(It is worth noting the difference between these two chain rules. In the first one, both f and g were functions of a *complex* variable, while in the second one f is a function of a complex variable but γ is a function only of a *real* variable. We have been told many times – and will shortly begin to see for ourselves! – that the requirement that a function of a complex variable have a derivative is far more restrictive than the requirement that a function of a real variable have a derivative: note that the difference is between the *domains*, and *not* the ranges. In other words, the difference is between a function *defined* on the complex numbers, and a function *defined* on the real numbers, and not a function taking values in the complex numbers and a function taking values in the real numbers.)

21. The Cauchy integral theorem, full proof. Recall that in section 19 above we showed that, if f is an analytic function on a simply-connected region, and C is any simple (non self-intersecting) closed curve contained in that region, then *if f has continuous first-order partial derivatives on the region,*

$$\int_C f(z)\,dz = 0.$$

We will now show that this result holds *without* the assumption of continuous first-order partial derivatives, which we will actually be able ultimately (next week) to derive as a consequence. Our treatment follows very closely that given in Goursat, §28.

Thus, let f be an analytic function on some region, and let C be any simple closed curve in that region such that f is analytic everywhere on the interior of C. Let U denote the region bounded by C, which is necessarily simply-connected; then by assumption f is analytic on U and on C. Now suppose that we subdivide U into squares and partial squares by drawing a square grid across it (see Figure 13 in Goursat for an example of what we mean by this). We let γ_k denote the boundary curve – oriented counterclockwise – of the kth full square, and γ_j' denote the boundary curve – again oriented counterclockwise – of the jth partial square. Then we claim that

$$\sum_k \int_{\gamma_k} f(z)\,dz + \sum_j \int_{\gamma_j'} f(z)\,dz = \int_C f(z)\,dz.$$

This is clear after a bit of thought, since the sides of the grid squares appear exactly twice, and in opposite directions, in the sum of integrals on the left, and hence cancel, meaning that we are left only with the integral around the boundary curve, i.e., the right-hand side.

We now claim that each of the integrals in the above sums is small. To see this, note that since the functions $z \mapsto a$ and $z \mapsto a(z - z_0)$ are analytic with continuous partial derivatives, the result from section 19 can be applied to show that around any closed curve they integrate to zero. (Another way of showing this, without applying Green's theorem – which we did in section 19 – is outlined in section 28 of Goursat.) Now consider $\int_{\gamma_k} f(z)\,dz$, and let z_0 be some point either inside or on γ_k. Then since f is analytic on U, we can write, for any point z on γ_k,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z - z_0)(z - z_0),$$

where $\epsilon(z - z_0) \to 0$ as $z \to z - z_0$. (In o notation, $\epsilon(z - z_0) = o(z - z_0)/(z - z_0)$, but we stick with this notation here for consistency with the lecture.) Now the functions $z \mapsto az$ and $z \mapsto a(z - z_1)$, where $a, z_1 \in \mathbf{C}$ are any two constant complex numbers, are both analytic with continuous first-order partials (this is entirely trivial!); thus the result from Section 19 shows that both integrate to zero around any simple closed curve. Hence we may write

$$\int_{\gamma_k} f(z)\,dz = \int_{\gamma_k} f(z_0) + f'(z_0)(z - z_0) + \epsilon(z - z_0)(z - z_0)\,dz = \int_{\gamma_k} \epsilon(z - z_0)(z - z_0)\,dz,$$

and similarly, letting z_0' denote some point within or on γ_j', and $\epsilon'(z - z_0')$ the corresponding function analogous to $\epsilon(z - z_0)$,

$$\int_{\gamma_j'} f(z)\,dz = \int_{\gamma_j'} \epsilon'(z - z_0')(z - z_0')\,dz.$$

Now recall that, if we have a curve γ of length ℓ and an analytic function f which is bounded by M on γ, then we have the bound

$$\left|\int_\gamma f(z)\,dz\right| \le \ell M.$$

Let us apply this to the two integrals above. Suppose that $\epsilon(z - z_0) \le \eta$ on γ_k, and that γ_k is a square with side lengths ℓ_k; then the total length of γ_k is $4\ell_k$, and moreover the function $z - z_0$ on γ_k is bounded by $\ell_k\sqrt{2}$ (since this is the length of a diagonal of γ_k and that is the farthest apart any two points can be on a square). Thus we may write

$$\left|\int_{\gamma_k} f(z)\,dz\right| \le 4\ell_k \cdot \eta\ell_k\sqrt{2} = 4\sqrt{2}\ell_k^2\eta = 4\sqrt{2}A_k\eta,$$

where $A_k = \ell_k^2$ is the area enclosed by γ_k. Similarly, suppose that $\epsilon'(z - z_0')$ is bounded by some number η' on γ_j'. Now γ_j' consists of parts of four sides of a square, together with some portion of C; thus, if we let ℓ_j' denote its side length and λ_j the length of that portion of C, then the length of γ_j' is bounded by $4\ell_j' + \lambda_j$. (This may be a very bad upper bound, since we may only have a small portion of the square sides, but the point is that it *is* an upper bound, and as we shall see later, it is a sufficiently good upper bound.) Now because we have decomposed the region U along a square grid, the region enclosed by γ_j' is a portion of a square, i.e., it is a region entirely contained in one of these squares; thus as before the function $z - z_0'$ on γ_j' is bounded by $\ell_j'\sqrt{2}$ and we may write

$$\left|\int_{\gamma_j'} f(z)\,dz\right| \le (4\ell_j' + \lambda_j) \cdot \eta'\ell_j'\sqrt{2} = (4A_j' + \ell_j'\lambda_j)\sqrt{2}\eta'.$$

Now we come to a technical point which is addressed in §29 of Goursat but which we shall just touch on without giving a formal proof. We know that as $z \to z_0$, $\epsilon(z - z_0) \to 0$, and similarly that $\epsilon'(z - z_0') \to 0$ as $z \to z_0'$. Similar relations will be true in all of the other squares and partial squares into which we have subdivided U.[3] This means that, by taking each individual square small, we can make the quantities η and η' small. We claim that by taking the entire *grid* arbitrarily fine, i.e., to have squares and partial squares which are arbitrarily small, *all* of the functions $\epsilon(z - z_0)$ and $\epsilon'(z - z_0)$, for *all* indices k and j (respectively), can *simultaneously* be made arbitrarily small. This does not automatically follow from the foregoing, but as it does seem reasonable, and the proof is slightly technical, we shall assume its truth and see how it can be used to derive the result. (As mentioned, an explanation of this result is given in §29 of Goursat for those who are interested.) Thus we assume that, for any $\eta_0 > 0$, by taking the grid sufficiently fine, we may assume that for all k and j, we may take $\eta, \eta' < \eta_0$. Now consider such a sufficiently fine grid, and let L be the side length of the squares in the grid; then we may write

$$\left|\sum_k \int_{\gamma_k} f(z)\,dz\right| \le \sum_k \left|\int_{\gamma_k} f(z)\,dz\right| \le 4\sqrt{2}\eta_0 \sum_k A_k \le 4\sqrt{2}\eta_0 A,$$

[3] Note that the points z_0 and z_0' actually depend on the indices k and j, respectively, but we have chosen not to indicate this in our notation just for simplicity.

where A denotes the area of some circle completely containing U, and such that all squares in the grid which intersect U are completely contained in that circle; similarly, letting λ denote the length of the curve C,

$$\left|\sum_j \int_{\gamma_j'} f(z)\,dz\right| \le \sum_j \left|\int_{\gamma_j'} f(z)\,dz\right| \le \sqrt{2}\left[4\sum_j A_j' + L\sum_j \lambda_j\right]\eta_0$$
$$\le \sqrt{2}(4A + L\lambda)\eta_0.$$

Thus, finally, we have

$$\left|\int_C f(z)\,dz\right| \le (4\sqrt{2}A + 4\sqrt{2}A + \sqrt{2}L\lambda)\eta_0 = (8\sqrt{2}A + L\sqrt{2}\lambda)\eta_0,$$

where η_0 is an arbitrary positive number. Now if we take any grid *finer* than the one we just considered, clearly L will decrease, while we can use the same A as before; in other words, if $\eta_0' < \eta_0$ and we consider any grid fine enough to have $\eta, \eta' < \eta_0'$, we may still write

$$\left|\int_C f(z)\,dz\right| \le (8\sqrt{2}A + L\sqrt{2}\lambda)\eta_0',$$

where A and L have the same values as they did before. By taking η_0' arbitrarily small, we see that the left-hand side must be arbitrarily small; since it does not depend on the grid, or η_0', it must actually be zero. This proves that

$$\int_C f(z)\,dz = 0,$$

as claimed.

In the above we have assumed that the function f was defined and analytic on a larger region completely containing the curve C and its interior. It turns out that one only need assume f to be analytic on the interior of C and *continuous* up to the boundary; a brief discussion of this is given in the footnote in Goursat, pp. 48 – 49 (of the typescript; p. 71 of the original).

22. Antiderivatives and branch cuts. Recall that in multivariable calculus we learned that a vector field $\mathbf{F}$ which is *conservative*, in the sense that its integral around any closed curve is zero, has a *potential function*, i.e., that there is a function f such that $\mathbf{F} = \nabla f$. Moreover, f can be constructed as

$$f(x,y) = \int_{(x_0,y_0)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r},$$

where (x_0, y_0) is any fixed point, and the line integral does not depend on the choice of curve from (x_0, y_0) to (x, y) since $\mathbf{F}$ is conservative. We now show a similar result in the case of analytic functions of a complex variable, though as usual the import is quite a bit deeper.

Thus suppose that f is a function analytic on a simply-connected region, pick some point z_0 in that region, and define a function

$$F(z) = \int_{z_0}^{z} f(z')\,dz'.$$

Let us see in what sense this formula defines a function. Recall that a function consists of three things: a domain, a range, and a rule giving an element of the range for any element of the domain. Here the domain can clearly be taken to be the simply-connected region on which f is analytic, and as usual we don't really worry about the range (F will certainly be in $\mathbf{C}$, at any rate). Thus we only need to consider in what sense the function above defines a rule which gives a complex number given any complex number in its domain. In order to evaluate the integral, we need to choose a particular path γ from z_0 to z. Suppose that γ_1 and γ_2 are two distinct paths from z_0 to z. If γ_1 and γ_2 have no intersection points other than their endpoints z_0 and z, then by running γ_1 forwards and γ_2 backwards we obtain a simple closed curve; if we call it γ, then we have

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz;$$

but by the Cauchy integral theorem, the left-hand side is zero, so that $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ and the integral evidently does not depend on the choice of curve in this case. It can be shown that this holds true even if the two curves have other intersection points; thus in the situation we are considering here, $F(z)$ depends only on the endpoints z and z_0, and not on the curve chosen from z_0 to z. It therefore does indeed give a single-valued function on the region.

Let us see whether we can compute its derivative. Thus we consider the quotient $[F(z+h) - F(z)]/h$. Now by choosing the curve used to calculate $F(z+h)$ so that it passes through z, we may write

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_z^{z+h} f(z')\,dz'.$$

Now we note that $\int_z^{z+h} dz' = h$, just as in elementary calculus on the real line (this can be shown by parameterising the line from z to $z+h$, for example); thus this last expression is equal to

$$\frac{1}{h}\int_z^{z+h} f(z') - f(z)\,dz'.$$

But now if h is very small, $f(z') - f(z)$ will be very small for all points on the straight line from z to $z+h$, which means that also $|f(z') - f(z)|$ will also be very small there; if η is any upper bound on this quantity, then we may write

$$\left|\frac{1}{h}\int_z^{z+h} f(z') - f(z)\,dz'\right| \leq \frac{1}{|h|}|h|\eta = \eta,$$

which means that by taking h sufficiently small, the above quantity must be less than η. But if we unravel everything, this means that the limit

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} - f(z)$$

must be zero, which means that F is analytic and $F'(z) = f(z)$, as we might have expected.

It is worth noting that, if z_1 and z_2 are any two complex numbers in the region above, then by taking the curve from z_0 to z_2 to pass through z_1, we may write

$$\int_{z_1}^{z_2} f(z')\,dz' = \int_{z_0}^{z_2} f(z')\,dz' - \int_{z_0}^{z_1} f(z')\,dz' = F(z_2) - F(z_1),$$

which shows that the fundamental theorem of calculus is true in this case as well.

Let us now consider what could have gone wrong if the region on which f was known to be analytic had not been simply connected. For the kinds of regions we are interested in here (essentially, open sets in the plane), the notion of 'simply connected' is a *global* notion, in the sense that it is in general a property of the entire region, not just some portion of the region. Alternatively, any region is *locally* simply connected, since if we consider any point in the region, there is certainly a small disk around that point contained in the region, and that disk will be simply connected. On such a disk, the above logic goes through, and thus we see that, at least near any given point, we can still construct an antiderivative of f in exactly the same fashion as above. What goes wrong is when we try to push this construction further away from the point. Thus suppose for example that f is analytic everywhere except at some point ζ_0, and let $z_0 \neq \zeta_0$; then near z_0 the function

$$F(z) = \int_{z_0}^{z} f(z')\,dz'$$

will be well-defined and independent of the curve connecting z_0 and z, and will give an antiderivative of f. But now consider trying to determine this function everywhere on some circle starting at z_0 which encloses the point ζ_0. At z_0 we have $F(z_0) = 0$ by definition. But when we traverse this circle around ζ_0, as we come back close to z_0, $F(z)$ may not be small, since there is no guarantee that the integral around the entire curve will vanish. This means that the limit of $F(z)$ may not equal 0 as $z \to z_0$ along this direction, and hence that it may not be possible to find a single-valued continuous antiderivative of f everywhere on the region. This is, in fact, a generalisation of what we have seen goes wrong when we consider the logarithm: since $\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}$, $\operatorname{Log} z$ is an antiderivative of $\frac{1}{z}$, and as we try to take its value along some closed curve containing the origin, we know that we run into problems of discontinuity or multivaluedness exactly like those just discussed. One solution to this problem in the general case is to use the solution we used for the logarithm, and take a branch cut starting at ζ_0 and going to infinity; the resulting region will be simply connected, and thus on it we may define a single-valued, continuous antiderivative using the above formula.

The notions above of starting out with an analytic function only defined on a small disk and attempting to extend it further are related to notions of *analytic continuation* which we shall discuss later on in the course.

23. An extension of Cauchy's integral theorem to non-simply connected regions. It turns out that there is a way of extending Cauchy's integral theorem to non-simply connected regions, in quite the same way one extends Green's theorem to such regions, which will be important to our derivation of the Cauchy integral formula and is also noteworthy in its own right. Suppose for definiteness that a function f is analytic everywhere on a region except at two holes (these could be two isolated points, or larger holes), and consider $\int_C f(z)\,dz$, where C is some simple closed curve in this region. As long as C does not enclose either of the holes, this integral will still vanish by the Cauchy integral theorem. Now if C contains just one of the holes, we may shrink it down to either the boundary curve of the hole (if the hole is itself a region) or to an arbitrarily small circle around the hole (if the hole is a point), and the integral of f over this new curve, call it C', will be equal to that of f over C: to see this, think of taking a point on C and joining it to some point on C' by a straight line; if we break this straight line open slightly, and pull the two edges apart, we will get a simple closed curve which does not enclose any singularities of f, and the integral over this curve will therefore vanish; but in the limit as the two lines come together, the integral over this curve is just

$$\int_C f(z')\,dz' - \int_{C'} f(z')\,dz',$$

assuming that we orient both C and C' counterclockwise. Thus these two integrals must be equal, as claimed.

In the case that C is a curve enclosing both holes, we may do something similar except that we will find

$$\int_C f(z')\,dz' = \int_{C_1'} f(z')\,dz' + \int_{C_2'} f(z')\,dz',$$

where C_1' and C_2' are curves enclosing the two holes, as described above. Here we are still assuming that all three curves are oriented counterclockwise. If we instead orient C_1' and C_2' clockwise, and call the resulting curves C_1 and C_2, then the above result becomes

$$\int_C f(z')\,dz' + \int_{C_1} f(z')\,dz' + \int_{C_2} f(z')\,dz' = 0,$$

i.e., the integral of f is still zero as long as we include curves around the singularities of f as well.

Summary:

- We derive the Cauchy integral formula from the Cauchy integral theorem for non-simply connected regions.
- We then proceed to show how it may be applied to derive Taylor and Laurent series expansions, and give a simple example.

(Goursat, §§33, 35, 37.)

24. Cauchy integral formula. Suppose that a function f is analytic everywhere inside a simple closed curve C, and continuous on C. Then from our comment at the end of §21 above it follows that the Cauchy integral theorem applies and we have

$$\int_C f(z)\,dz = 0.$$

Now let us fix some point z_0 in the *interior* of the curve C. Then the function

$$\frac{f(z)}{z - z_0}$$

is clearly analytic everywhere inside C except at the point z_0. If we let C' be a small circle centred at z_0 and contained in the interior of C, say with radius $r > 0$, oriented counterclockwise, then by the discussion and result in §23 above we have

$$\int_C \frac{f(z)}{z - z_0}\,dz = \int_{C'} \frac{f(z)}{z - z_0}\,dz;$$

in other words, we are able to replace the (fairly arbitrary and possibly very complicated) curve C by the (presumably much simpler) curve C'. Now we can make C' as small as we like, and the above result will still hold, since $z = z_0$ is the only point inside C at which the integrand $f(z)/(z - z_0)$ is not analytic. Now f is analytic at z_0, so near z_0 we can write as we have before

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z - z_0)(z - z_0),$$

where $\epsilon(z - z_0) \to 0$ as $z \to z_0$. Thus we may write

$$\begin{aligned}\int_{C'} \frac{f(z)}{z - z_0}\,dz &= \int_{C'} \frac{f(z) - f(z_0)}{z - z_0} + \frac{f(z_0)}{z - z_0}\,dz \\ &= \int_{C'} f'(z_0) + \epsilon(z - z_0) + \frac{f(z_0)}{z - z_0}\,dz. \qquad (1)\end{aligned}$$

The integral of $f'(z_0)$ over C' is clearly zero since $f'(z_0)$ is a constant; we shall show in a moment that the integral of $\epsilon(z - z_0)$ over C' must be zero also. Thus we consider the integral

$$\int_{C'} \frac{f(z_0)}{z - z_0}\,dz.$$

Now C' is a circle of radius r centred at z_0, and can be parameterised as

$$z(t) = z_0 + re^{it}, \qquad t \in [0, 2\pi],$$

whence the integral above becomes[1]

$$\int_0^{2\pi} \frac{f(z_0)}{re^{it}} rie^{it}\,dt = \int_0^{2\pi} if(z_0) = 2\pi i f(z_0).$$

[1] Note that this is not really just a 'substitution' as used in elementary calculus; most obviously, substitution in elementary calculus was only shown for integrals of functions of a real variable, and here we are dealing with functions of a complex variable. More substantively, though, the process by which we reduce a contour integral to a definite integral in terms of a parameterisation of the curve follows from the *definition* of the contour integral as we showed above. The formal similarity is however obvious and worth noting as an aid to memory, though it should be borne in mind that the two processes are not identical.

Note that this does not depend on the radius r. Now, finally, consider the integral

$$\int_{C'} \epsilon(z - z_0)\, dz.$$

To evaluate it, note that since $\epsilon(z - z_0) \to 0$ as $z \to z_0$, by taking r sufficiently small we may assume that $|\epsilon(z - z_0)| < 1$ on C'; thus the absolute value of the above integral satisfies

$$\left| \int_{C'} \epsilon(z - z_0)\, dz \right| \leq 2\pi r;$$

thus if we take the limit as $r \to 0$ this integral must vanish. Now if we investigate equation (1), we find that $\int_{C'} \epsilon(z - z_0)\, dz$ is the *only* term in the whole equation which could depend on r; thus *it* can't depend on r either, so since its limit as $r \to 0$ must vanish, it must actually be zero for all r (all r sufficiently small that C' lies entirely inside C, anyway!). Putting all this together, we obtain finally

$$\int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0),$$

or

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz. \tag{2}$$

This is called the *Cauchy integral formula*. Thus the *Cauchy integral theorem* tells us that the integral of an analytic function around a closed curve is 0, while the *Cauchy integral formula* gives us a formula for calculating the *value* of an analytic function inside some curve in terms of an integral around that curve.

Let us expand on this last point for a bit. In equation (2), z_0 is *any* point inside the curve C. Note though that the right-hand side of the equation depends *only* on the values of f *on* the curve C! In other words, what we have here is a formula which will give us the value of a function at any point inside a curve, given only its values on that curve. In the one-variable case, this would be equivalent to saying that the values of a function at the endpoints of an interval determine the function everywhere inside the integral, a claim so patently false as to be silly. For those of you who have seen some partial differential equations, this property should be reminiscent of the solution to boundary-value problems, particularly for Laplace's equation: there, in fact, if one has a Green's function, one can actually produce an integral formula quite reminiscent of (3) for the value of the solution inside a region given only its values on the boundary of the region.[2]

Let us rewrite equation (3) as

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z}\, dz', \tag{3$'$}$$

to emphasise that what we have on the left-hand side is actually a full function rather than a single value. Now it can be shown (see Goursat, §33) that we can differentiate the right-hand side by taking the derivative under the integral sign. In other words, since the point z in (3') must lie *within* C, it cannot lie on C, so that the quantity z' in the integrand is never equal to z and we may therefore write for every z' on C

$$\frac{d}{dz} \frac{1}{z' - z} = \frac{1}{(z' - z)^2},$$

by the power rule and chain rule for differentiating functions of a complex variable. (Note that, while in the integrand we view $1/(z' - z)$ as a function of z', with z fixed, here we view it as a function of z with z' fixed.) Now assuming that we can differentiate under the integral sign, we may write

$$f'(z) = \frac{d}{dz} \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z}\, dz' = \frac{1}{2\pi i} \int_C \frac{f(z')}{(z' - z)^2}\, dz'.$$

[2] Note that there are some connections between these last two sentences. A harmonic function of a single variable would be an f which satisfied the equation $f'' = 0$; the only solutions to this equation are functions $f(x) = ax + b$, where a and b are constants – and a little thought shows that *these* functions actually *do* satisfy the property just stated: in other words, they *are* determined by their values on the endpoints of any interval! The class of harmonic functions on the line, though, is too small to be very interesting.

Assuming that we may again differentiate under the integral sign, we see that the right-hand side also has a derivative and in fact, since

$$\frac{d}{dz}\frac{1}{(z'-z)^2} = \frac{2}{(z'-z)^3},$$

this derivative is

$$\frac{d}{dz}\frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-z)^2}\,dz' = \frac{1}{2\pi i}\int_C \frac{2f(z')}{(z'-z)^3}\,dz'.$$

Continuing in the same way, then, we may evidently write

$$\frac{d^n}{dz^n}\frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\int_C \frac{n!f(z')}{(z'-z)^{n+1}}\,dz'.$$

Since the integral we are differentiating above is equal to $f(z)$, this shows that $f(z)$ has *arbitrarily many* derivatives, as we have often claimed and never actually proved until now. Note that the only assumption we needed to make was that f be analytic on a certain region; we did *not* need to assume that the derivative of f was continuous, or that the real and imaginary parts of f had continuous partial derivatives. These results now follow as a consequence, since the derivative of f must itself have a derivative, and hence must be analytic, hence continuous, showing that the real and imaginary parts of f do indeed have continuous partial derivatives.

To sum up, then, we have, for any nonnegative integer n, the *Cauchy integral formula*

$$f^{(n)}(z) = \frac{1}{2\pi i}\int_C \frac{n!f(z')}{(z'-z)^{n+1}}\,dz'.$$

Let us give a couple examples.

EXAMPLES. If $f(z) = a$ is some constant, then we have

$$a = f(z) = \frac{1}{2\pi i}\int_C \frac{a}{z'-z}\,dz',$$

i.e., that if z is any point inside the simple closed curve C, then $\int_C \frac{1}{z'-z}\,dz' = 2\pi i$; this is a result worth remembering by itself. Now since f is constant, we must have $f'(z) = 0$, and hence $f^{(n)}(z) = 0$ for all $n \geq 1$; the above formula then gives

$$0 = f^{(n)}(z) = \frac{1}{2\pi i}\int_C \frac{a}{(z'-z)^{n+1}}\,dz',$$

which gives

$$\int_C \frac{1}{(z'-z)^{n+1}}\,dz' = 0$$

whenever z' is inside the simple closed curve C and $n \geq 1$. Note that this does *not* follow from the Cauchy integral theorem since the integrand here is *not* analytic within the curve C. Thus we have an extension of the Cauchy integral theorem in this case. Again, this result is worth remembering all by itself.

25. Taylor series. Now that we know that any analytic function must have arbitrary many derivatives, we know that we can formally write out its Taylor expansion

$$\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(z-a)^n, \tag{4}$$

where a is any point in the region on which f is analytic. The existence of the derivatives of f, though, does not prove that this series actually converges to f anywhere except at $z = a$ (where it does trivially since by convention the series above is simply $f(a)$ when $z = a$). Here we shall derive the Taylor expansion by a different method, namely as an application of the Cauchy integral formula. Our exposition closely follows that of Goursat, §35.

Since the series in (4), if it converges anywhere except at $z = a$, must converge on a disk centred at a, let us take our curve C to be a circle of radius R centred at a. Now for any z inside C we have the Cauchy integral formula for f:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz'.$$

We shall show how to expand $\frac{1}{z'-z}$ in a power series. We have

$$\frac{1}{z'-z} = \frac{1}{(z'-a)-(z-a)} = \frac{1}{z'-a}\frac{1}{1-\frac{z-a}{z'-a}}; \tag{5}$$

factoring out $z' - a$ like this is legitimate since here we are only concerned with the expression $1/(z' - z)$ when z' is a point *on* the curve C, and the point a is inside the curve. In fact, in this case, since the curve C is a circle of radius R centred at a, we actually have $|z' - a| = R$. Suppose that $|z - a| = r$; since z also lies inside C, we must have $r < R$. Now we would like to expand the second term in (5) above in a series. We shall augment our treatment in the lecture by providing a careful proof. (Our treatment in the lecture corresponded to taking $N \to \infty$ immediately and dropping the remainder terms, namely those terms coming from w^{N+1} below.) Recall the geometric series

$$\sum_{n=0}^{N} w^n = \frac{1-w^{N+1}}{1-w},$$

which is valid for any complex number $w \neq 1$;[3] from this we have

$$\frac{1}{1-w} = \sum_{n=0}^{N} w^n + \frac{w^{N+1}}{1-w}.$$

In our case, this gives from (5)

$$\begin{aligned}\frac{1}{z'-z} &= \frac{1}{z'-a}\left[\sum_{n=0}^{N}\left(\frac{z-a}{z'-a}\right)^n + \frac{1}{1-\frac{z-a}{z'-a}}\left(\frac{z-a}{z'-a}\right)^{N+1}\right] \\ &= \sum_{n=0}^{N}\frac{(z-a)^n}{(z'-a)^{n+1}} + \frac{1}{z'-z}\left(\frac{z-a}{z'-a}\right)^{N+1}.\end{aligned}$$

Substituting this back in to (4), we see that

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\int_C \sum_{n=0}^{N}(z-a)^n\frac{f(z')}{(z'-a)^{n+1}} + \frac{f(z')}{z'-z}\left(\frac{z-a}{z'-a}\right)^{N+1}\,dz' \\ &= \frac{1}{2\pi i}\left[\sum_{n=0}^{N}(z-a)^n\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz'\right] + \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\left(\frac{z-a}{z'-a}\right)^{N+1}\,dz'.\end{aligned} \tag{6}$$

Let us consider the last term above. Since f is continuous on C, it must be bounded on C; let $M > 0$ be such that $|f(z')| < M$ when z' is on the curve C. Now since $|z' - a| = R$ and $|z - a| = r < R$, we see that $|z' - z| \geq R - r$ (this is just the triangle inequality $|z' - a| \leq |z' - z| + |z - a|$); thus

$$\left|\frac{1}{z'-z}\right| = \frac{1}{|z'-z|} \leq \frac{1}{R-r}.$$

[3] In fact, this formula is valid in any *ring* as long as $1 - w$ is invertible in that ring; i.e., it is a purely algebraic result.

Further,

$$\left|\frac{z-a}{z'-a}\right|^{N+1} = \left(\frac{r}{R}\right)^{N+1}.$$

Thus the absolute value of the second term can be bounded as follows:

$$\left|\frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\left(\frac{z-a}{z'-a}\right)^{N+1} dz'\right| \leq \frac{1}{2\pi}\cdot 2\pi R\cdot M\cdot \frac{1}{R-r}\cdot\left(\frac{r}{R}\right)^{N+1} = \frac{MR}{1-R}\left(\frac{r}{R}\right)^{N+1}.$$

Since $r < R$, this quantity must go to zero in the limit as $N \to \infty$; substituting this into (6) gives

$$\frac{1}{2\pi i}\left[\sum_{n=0}^{\infty}(z-a)^n \int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz'\right] = f(z) - \lim_{N\to\infty}\frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\left(\frac{z-a}{z'-a}\right)^{N+1} dz' = f(z),$$

or to write it out more clearly,

$$f(z) = \sum_{n=0}^{\infty}(z-a)^n \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz'.$$

But by the Cauchy integral formula for $f^(n)$, the integral here is simply $\frac{1}{n!}f^{(n)}(a)$, and we have thus proven the Taylor series expansion for f,

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(z-a)^n,$$

which will be valid on any disk centred at a on which f is analytic. Note that the above argument shows quite rigorously both that the above series converges and that it converges to $f(z)$, given only the general Cauchy integral formula. So if you had never seen a proof that a Taylor series converges to the function it comes from, now you have!

26. Laurent series. It turns out that for many applications it is important to be able to treat functions which have varies kinds of *singularities*, i.e., which fail to be analytic at various points or regions of the plane. While such functions will still clearly have Taylor series expansions on any disk not containing any of these singularities, it turns out to be useful to consider a more general type of expansion which will represent the function on a region surrounding the singularities. These are called *Laurent series.*

Thus suppose that we have a function f which is analytic on an *annulus*; specifically, suppose that C and C' are two circles, centred at a point a, with radii R and R' respectively, where $R > R'$ (so that C' is the inner circle), and both oriented counterclockwise, and that f is analytic on the region between C and C'. We shall extract a series expansion for f from the general Cauchy integral theorem in the same way we found the Cauchy integral formula and then used it to extract the Taylor expansion for f in the previous two sections. Our first step is thus to produce a generalisation of the Cauchy integral formula to the present case. The generalisation is not at all hard. Let z be any point in the annulus between C and C', and let γ be a small circle centred at z and with radius r, oriented counterclockwise and entirely contained in the region between C and C'. Then by the general Cauchy integral theorem in §23, we have

$$\int_C \frac{f(z')}{z'-z}\,dz' = \int_{C'} \frac{f(z')}{z'-z}\,dz' + \int_\gamma \frac{f(z')}{z'-z}\,dz'.$$

Now since γ is entirely contained in the region between C and C', f must be analytic everywhere on and inside γ, which means that by the usual Cauchy integral formula the second integral above is just

$$\int_\gamma \frac{f(z')}{z'-z}\,dz' = 2\pi i f(z),$$

and the above formula gives

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\int_{C'} \frac{f(z')}{z'-z}\,dz';$$

in other words, we can generalise the Cauchy integral formula to the case of a function analytic *between* two curves if we integrate over both of them with the correct orientation (equivalently, including the correct minus sign). Evidently we could also extend the formula to a situation where a function was analytic on a region with *multiple* holes, but we do not need that here.

Now the first integral above can be treated exactly as before, giving ultimately

$$\sum_{n=0}^{\infty}(z-a)^n \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz',$$

but note that in this case we *cannot* replace the integral with $f^{(n)}(a)/n!$, since f is not known to be analytic at a (f might not even be defined at a, for that matter!). The second integral can be treated by slightly adapting this method. Since in the second integral the point z' lies on C', letting $|z-a| = r$ we have $|z'-a| = R' < r$; thus we may write

$$-\frac{1}{z'-z} = \frac{1}{z-z'} = \frac{1}{(z-a)-(z'-a)} = \frac{1}{z-a}\frac{1}{1-\frac{z'-a}{z-a}};$$

thus we have an analogue to formula (6) but with z' and z interchanged except inside f:

$$\begin{aligned}-\frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z'-z}\,dz &= \frac{1}{2\pi i}\left[\sum_{n=0}^{N}(z'-a)^n\int_{C'}\frac{f(z')}{(z-a)^{n+1}}\,dz\right] + \frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z-z'}\left(\frac{z'-a}{z-a}\right)^{N+1}dz\\ &= \frac{1}{2\pi i}\left[\sum_{n=0}^{N}\frac{1}{(z-a)^{n+1}}\int_{C'}f(z')(z'-a)^n\,dz'\right] + \frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z-z'}\left(\frac{z'-a}{z-a}\right)^{N+1}dz'.\end{aligned}$$

Since we now have, as just noted, $|z'-a| = R' < r = |z-a|$, the argument given above shows that the second integral vanishes in the limit as $N \to \infty$, and we obtain the series expansion

$$-\frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z'-z}\,dz = \sum_{n=0}^{\infty}\frac{1}{(z-a)^{n+1}}\int_{C'}(z'-a)^n f(z')\,dz'.$$

Thus, finally, we find that $f(z)$ can be expressed as the sum of two series:

$$f(z) = \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{C'}\frac{f(z')}{(z'-a)^{n+1}}\,dz' + \sum_{n=0}^{\infty}\frac{1}{(z-a)^{n+1}}\int_{C'}(z'-a)^n f(z')\,dz'.$$

To simplify this a bit, let us make the definitions

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz' \quad (n \geq 0), \qquad b_n = \frac{1}{2\pi i}\int_{C'}(z'-a)^{n-1}f(z')\,dz', \quad (n \geq 1)$$

where in b_1 we have $(z'-a)^0 = 1$ since $z' \neq a$, as z' is on C' and a is inside C'. Then we can write

$$f(z) = \sum_{n=0}^{\infty}a_n(z-a)^n + \sum_{n=1}^{\infty}b_n\frac{1}{(z-a)^n};$$

in other words, whereas in the previous section, when f was analytic everywhere inside the circle C and we could write it as a sum of powers of $z-a$, in the case when f is analytic only on an annular region, we must write f as an infinite series of powers of $z-a$ *and* $1/(z-a)$. This is reasonable since $1/(z-a)$ will not be analytic at $z = a$; but note that f may be singular at other points inside C' than just a.

Before ending with an example, it is probably worthwhile to step back a bit and consider what be the importance of the results we have derived in the last three sections. As a concise summary, and for comparison, these are

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz',\\
f(z) &= \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz',\\
f(z) &= \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{C'} \frac{f(z')}{(z'-a)^{n+1}}\,dz' + \sum_{n=0}^{\infty}\frac{1}{(z-a)^{n+1}}\int_{C'}(z'-a)^n f(z')\,dz',
\end{aligned}$$

where f is assumed to be analytic within the arbitrary simple closed curve C in the first line, within the circle C in the second, and between the circles C' and C in the third. All three of these are *representation formulæ*; i.e., they give $f(z)$ as a special type of expression (an integral in the first case, series in the latter two). One of the uses of formulæof this sort is that they give us concrete ways of writing out f, which allow us to perform certain manipulations which would be much harder without them. Another, slightly more abstract, perspective is that these formulægive us a way of breaking f down into other data, which may encode the information we need for a specific problem in a more convenient way than the map $z \mapsto f(z)$ all by itself. For example, if we are only interested in knowing $f(1)$, then the simpler the formula for f the better; but if we are interested in knowing $\int_{C'} f(z)\,dz$, then the simpler the expression for b_1 the better.

On the other hand, these formulæare so general that it will require a fair bit more work before we get to the concrete applications in which they are so powerful. Thus unfortunately we shall have to stop at the vague indications in the previous paragraph for the time being, with a promise to say more about it later.

Let us do an example.

EXAMPLE. Let p be a positive integer, let $a \in \mathbf{C}$, and define the function f on $C\backslash\{a\}$ by

$$f(z) = \frac{1}{(z-a)^p}.$$

Then f is analytic everywhere on the plane except at $z = a$. (This kind of singularity, incidentally, is called a *pole* of order p; we shall study these systematically later.) Thus we expect to be able to expand f as a Laurent series. Actually it is quite obvious that $f(z)$ as given *is* a (single-term) Laurent series, so actually we already know this without any calculation; but let us work out the integrals anyway to see what happens. In this case we may take C and C' to be any circles centred at a, say with radii R and R', where the only condition on these radii is that $R > R'$. We have first of all

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz' = \frac{1}{2\pi i}\int_C \frac{1}{(z'-a)^{n+p+1}};$$

now $n \geq 0$, while $p \geq 1$, so $n + p \geq 1$ and by the example we did at the end of §24 above we must have $a_n = 0$. Similarly,

$$b_n = \frac{1}{2\pi i}\int_{C'}(z'-a)^{n-1}f(z')\,dz' = \frac{1}{2\pi i}\int_{C'}(z'-a)^{n-1-p}\,dz';$$

if $1 \leq n < p$ (note that if $p = 1$ there will not be any such n, but that doesn't matter) then we must have $n-1-p < -1$, so this integral is zero for the same reason. Now if instead we have $n > p$, then $n-1-p \geq 0$, so the integrand is actually analytic, and by the Cauchy integral theorem we have again $b_n = 0$. The only case left is $n = p$; in this case we have

$$b_p = \frac{1}{2\pi i}\int_{C'}\frac{1}{z'-a}\,dz' = 1,$$

by the first example at the end of §24 above. Thus we can write

$$f(z) = \sum_{n=0}^{\infty}a_n(z-a)^n + \sum_{n=1}^{\infty}b_n(z-a)^{-n},$$

where $a_n = 0$ for all n and $b_n = 0$ except for $n = p$, where $b_p = 1$. The series on the left thus trivially give $\frac{1}{(z'-a)^p}$, as they should.

For those of you who have seen orthogonal bases in vector spaces with an inner product, it is worth noting the formal similarity between the above procedure and that of determining components along the basis vectors in an orthonormal basis. We are not going to make this formal similarity precise, but it is worth noting anyway.

APPENDIX I. REVIEW OF MULTIVARIABLE CALCULUS AND LINEAR ALGEBRA

I. MULTIVARIABLE CALCULUS

1. Parametric curves. A *plane parametric curve*[1] is a curve in the plane which can be described by two equations

$$x = x(t), \qquad y = y(t), \qquad\qquad (t \in [a, b])$$

for some interval $[a, b]$; in other words, for every point (x, y) on the curve, there is some value $t \in [a, b]$ such that $x = x(t)$ and $y = y(t)$. (Note that this t need not be unique.) More informally, if we view t as a dynamical quantity, the point $(x(t), y(t))$ 'traces out' the entire curve as t varies from a to b. It is often convenient to represent the point $(x(t), y(t))$ by a single function, often called $\gamma(t)$ (the Greek letter gamma), so that $\gamma(t) = (x(t), y(t))$. We shall use γ (without t) to refer to the entire curve, considered as a single object. When necessary to distinguish between the *function* $\gamma(t)$ and the plane curve this function represents, we shall call the latter the *image* of γ.

A curve is called *closed* when (in the notation of the previous paragraph) $\gamma(a) = \gamma(b)$. A closed curve which does not intersect itself (i.e., for which the value of t mentioned above *is* unique) is called a *Jordan curve*. A Jordan curve γ is said to be oriented *counterclockwise* if, as t increases from a to b, the point $\gamma(t)$ traces out the curve in a counterclockwise direction, and similarly to be oriented *clockwise* if this point traces out the curve in a clockwise direction.[2] We note for future use that if D is a connected region of the plane, then its boundary curve is always a Jordan curve. This result has a converse in the so-called *Jordan curve theorem* which we shall mention later on in the course.

General parametric curves can display pathological behaviour, even when $x(t)$ and $y(t)$ are both continuous.[3] In this course we shall deal exclusively with so-called *piecewise-smooth curves*, defined as follows. A parametric curve γ is said to be *piecewise-smooth* on an interval $[a, b]$ if (i) it is continuous on $[a, b]$ and (ii) there are points $t_0 = a < t_1 < \cdots < t_n = b$ such that on each subinterval (t_i, t_{i+1}), $i = 0, \cdots, n-1$, the *derivative* $\gamma'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$[4] exists, and is continuous and nonzero. (Condition (ii) amounts to saying that $x(t)$ and $y(t)$ are continuously differentiable on (t_i, t_{i+1}), and that $x'(t)$ and $y'(t)$ never vanish simultaneously. This last requirement is necessary to avoid 'corners'; see the practice problems!)

A piecewise smooth curve has a well-defined length. Recall that the *length* of a parametric curve γ defined on some interval $[a, b]$ and such that γ' is continuous there is given by

$$\int_a^b |\gamma'(t)|\, dt,$$

where $|\cdot|$ denotes the length of a vector. This definition can be extended to a piecewise-smooth curve in an obvious way: if $t_0, t_1, \ldots, t_n$ are the points given in the definition of piecewise-smoothness, then we define the length of γ to be[5]

$$\int_{t_0}^{t_1} |\gamma'(t)|\, dt + \int_{t_1}^{t_2} |\gamma'(t)|\, dt + \cdots + \int_{t_{n-1}}^{t_n} |\gamma'(t)|\, dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |\gamma'(t)|\, dt.$$

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[1] Parametric curves can, of course, also be considered in three (and even arbitrary) dimensions. In this course, though, we shall only need them in two.

[2] Note that this definition would not make sense for a self-intersecting curve: for example, no matter how you trace out a figure-eight, the upper part will be oriented one way while the lower part will be oriented another.

[3] For example, one can find a parametric curve which – at least if we are allowed to replace the bounded interval $[a, b]$ by the whole real line – essentially fill out an entire two-dimensional region!

[4] While we shall not need to make this distinction in this course, it bears pointing out that, technically speaking, points and vectors are not identical, and when one must distinguish between them, a curve γ gives a point for each t while its derivative gives a vector.

[5] To be precise, the integrals here should be understood as improper integrals obtained by integrating from something slightly greater than t_i to something slightly less than t_{i+1}, and then taking the limit as these endpoints approach those two values, respectively; but this is generally not something which needs to be made explicit in practice, and we shall generally pass over it in silence in similar cases in the future.

The main use we shall make of parametric curves is in *line integrals* (see §3 below), and also in describing how two points in the plane are connected. This latter will become clearer as we progress through the course. The fact that two real numbers are essentially only connected in one way, while two complex numbers can be connected in multiple ways, some of which may be distinct (in an appropriate sense), is part of what makes complex analysis interesting.

2. Partial derivatives. Suppose that we have a function f defined on a region of the plane, which we suppose has Cartesian coordinates (x, y). We define its *partial derivatives* with respect to x and y to be

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h},$$
$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{(f(x, y+h) - f(x, y)}{h}.$$

We recall that in multivariable calculus we saw that the *existence* of both partial derivatives still allowed for quite a bit of pathological behaviour. It turns out that for functions of a complex variable there are additional requirements on the partial derivatives in order for the function to have a single *complex* derivative, and that these requirements, though simple, lead to far-reaching results which rule out all such pathological behaviours.

Recall that if a function f has a local extremum at a point where its partial derivatives exist, then they must both vanish.

Some examples of partial derivatives are given in the review problems.

[This paragraph is an aside for students who have had MAT237 or MAT257, or who have otherwise learned how to view the derivative as a linear map. In this class we shall be interested in complex-valued functions of a complex variable; since the set of complex numbers is a two-dimensional vector space over the real numbers, this means that we are in essence considering functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ (or, in essence, a vector field on $\mathbf{R}^2$). Thus the derivative of such a function, in the multivariable-calculus sense, should be a linear map from $\mathbf{R}^2$ to $\mathbf{R}^2$ approximating the original function at the point of differentiation. It turns out that the requirement that a complex derivative exists requires that this map be a composition of an isotropic scaling (i.e., multiplication by a single real number) and a rotation. This is the basis for the study of functions of a complex variable as *conformal maps*, namely functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ which preserve angles.]

3. Line integrals and vector fields. Suppose that γ is a piecewise-smooth curve on an interval $[a, b]$ ($\gamma(t) = (x(t), y(t))$), and that f is a continuous function defined on some set containing the image of γ. Then we define three different types of *line integral* along γ, as follows. Let $t_0, t_1, \cdots, t_n$ be the points given in the definition of piecewise-smoothness; then we define

$$\int_\gamma f\,dx = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(x(t), y(t))\,x'(t)\,dt$$
$$\int_\gamma f\,dy = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(x(t), y(t))\,y'(t)\,dt$$
$$\int_\gamma f\,ds = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(x(t), y(t))\,|\gamma'(t)|\,dt,$$

and call these *the line integrals of f along γ with respect to x, y, and arclength*, respectively.

Recall that a *vector field* on a region of $\mathbf{R}^2$ is a function which to every point in its domain associates a vector in $\mathbf{R}^2$; in other words, it can be written as a function $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, where $P(x, y)$ and $Q(x, y)$ are functions defined on the region called (naturally) the *components* of the vector field. If the vector field $\mathbf{F}$ is defined on a region containing γ, then we may combine the line integrals with respect to x and y of the components of $\mathbf{F}$ to define a new line integral, as follows:

$$\int_\gamma P(x, y)\,dx + \int_\gamma Q(x, y)\,dy = \int_\gamma \mathbf{F}(x, y) \cdot d\mathbf{x}.$$

We call this the line integral of the vector field along the curve γ. Recall the following *fundamental theorem of calculus for line integrals*: If $\mathbf{F} = \nabla f$ for some function f, i.e., if $\mathbf{F}$ is a gradient, then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = f(\gamma(b)) - f(\gamma(a)),$$

and this integral is therefore independent of the choice of path γ. This notion of *path-independence* (this is a standard term, though in our current setting it would be more natural to call it *curve-independence*!), namely that the line integral along a certain curve only depends on the end-points of the curve and not on the curve itself, is of central importance in the study of analytic functions of a complex variable. Recall that it is equivalent to the requirement that the line integral along any closed curve be zero. This is in turn related to *Green's theorem*, which states that for any vector field $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and any closed curve γ bounding a connected region D and oriented counterclockwise,

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where the latter is an *area integral* over the region D. This is a special case of *Stokes's theorem*, which we shall not need in its full generality but which we state here because it provides useful notation: If S is any (sufficiently smooth) connected surface in $\mathbf{R}^3$ with boundary curve C, and S and C are oriented consistently,[6] then

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dA.$$

Here the second integral is a *surface integral* and $\mathbf{n}$ represents the *unit normal* to the surface S, but we shall not need these things in this class. The *curl* of a vector field can be defined heuristically as $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$; if $\mathbf{F}$ is a vector field on $\mathbf{R}^2$ then the curl can be taken to be the single number

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

appearing in Green's theorem. For us this is the only case for which we shall need to use the curl (and we shall not need to use it much even here).

Note now that Green's theorem tells us that line integrals of a vector field are path-independent exactly when the curl of that vector field is zero. Such a vector field is called *conservative*, though we shall only need this term only occasionally. We have seen that a vector field which is the gradient of a function is conservative; on a so-called *simply connected region* – by which we mean a region 'without holes', or, more precisely, whose boundary is a *single* Jordan curve – the converse is also true. We shall see that these results have analogues in the theory of functions of a complex variable, though the results generally are not quite exact copies.

II. LINEAR ALGEBRA

4. Matrices. In this course we shall not need much from the results of linear algebra, but mostly a familiarity with its concepts. Recall that a *matrix of size m by n* is a two-dimensional array of numbers

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

and is called *square* if $m = n$. The *product* of matrices $[a_{ij}]$ and $[b_{jk}]$ of sizes m by n and n by ℓ is defined to be the matrix $[c_{ik}]$ of size m by ℓ given by

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

[6] For us, this just means that were γ oriented *clockwise* we would need to introduce an extra minus sign on the right-hand side of Green's theorem.

Recall that the *identity matrix of size n by n*

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

has the property that $AI = A$ and $IB = B$ for any matrices A and B of size m by n and n by ℓ, respectively. If a matrix $A = [a_{ij}]$ is square of size n by n, then its *inverse* (when it exists) is a matrix A^{-1} of size n by n satisfying

$$AA^{-1} = A^{-1}A = I.$$

In general, finding an inverse matrix is hard. For two-by-two matrices, however, there is a simple formula which is often useful, given by *Cramer's rule*: If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

as long as $ad - bc \neq 0$. The quantity $ad - bc$ is called the *determinant* of the matrix A; the notion of determinant can be defined for a square matrix of any size, but as the general definition is complicated and we shall not need it in this course we pass over it for the moment.

Recall that a matrix of size m by n can be viewed as giving a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$. In particular, a 2 by 2 matrix can be viewed as a linear transformation on the plane. Two particularly important and simple examples are *isotropic scaling* and *rotation.* The first is just multiplication by a single scalar and corresponds to the matrix ($\lambda \neq 0$)

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad \text{which has inverse} \qquad \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{bmatrix}.$$

The second is a bit more complicated. Consider rotation of the plane by an angle θ counterclockwise around the origin. Since vector addition and scalar multiplication in the plane can be defined in terms of geometric pictures which are transformed rigidly by such a rotation, we see that this rotation must be linear; thus it suffices to determine its effect on the basis vectors $\mathbf{i}$ and $\mathbf{j}$ of the plane. If we rotate the vector $\mathbf{i}$ by an angle θ counterclockwise around the origin, a little geometry makes it clear that we obtain the vector $\cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, while if we rotate $\mathbf{j}$ the same way we obtain the vector $-\sin\theta i + \cos\theta\mathbf{j}$; thus the matrix giving this transformation is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

We note two interesting properties of this matrix: first, its determinant is

$$\cos\theta \cdot \cos\theta - (-\sin\theta) \cdot \sin\theta = \cos^2\theta + \sin^2\theta = 1;$$

secondly, its inverse is (by the general formula above)

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

which is just the original matrix with θ replaced by $-\theta$! This makes good sense since the inverse to a counterclockwise rotation by θ is a clockwise rotation by θ, which is essentially just a counterclockwise rotation by $-\theta$.

APPENDIX II. BASIC DEFINITIONS AND PROPERTIES OF COMPLEX NUMBERS.

1. Basic definitions. A *complex number* is an abstract quantity $z = a + ib$, where a and b are real numbers and i is an abstract quantity which we require to satisfy $i^2 = -1$.[1] We require that these numbers satisfy all of the usual properties of arithmetic; thus if $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we have

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$
$$z_1 \cdot z_2 = a_1a_2 - b_1b_2 + i(a_1b_2 + b_1a_2.$$

We define the *conjugate* of a complex number $z = a + ib$ to be the complex number

$$\overline{z} = a - ib,$$

and note that the product

$$z\overline{z} = a^2 + b^2$$

is always real and nonnegative. By the Pythagorean theorem, $\sqrt{z\overline{z}}$ is the distance from the origin to the point (a, b), and we call this quantity the *modulus*[2] (or, sometimes, the *absolute value* or even the *length*) of the complex number z, and denote it as

$$|z| = \sqrt{z\overline{z}}.$$

The function $|\cdot|$ satisfies all of the usual properties of the absolute value function on real numbers:

$$|z_1z_2| = |z_1||z_2|, \qquad |z_1 + z_2| \leq |z_1| + |z_2|.$$

Since geometrically $|\cdot|$ represents distance from the origin, a set of the form

$$\{z||z - z_0| < R\}$$

is a circle of radius R centred at the point corresponding to z_0.

The ratio of two complex numbers can be determined as follows. Let $z_1 = a+ib$, $z_2 = c+id$ be complex numbers with $z_2 \neq 0$; then

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(a + ib)(c - id)}{c^2 + d^2}$$
$$= \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.$$

Let $z = a + ib$ be a complex number, and consider the corresponding point (a, b) in the plane. Let this point have a polar representation (r, θ), where r is the distance to it from the origin and θ is the angle from the positive x axis to a ray from the origin to the point, measured counterclockwise. Then we may write

$$a = r\cos\theta, \qquad b = r\sin\theta,$$

so that we have

$$z = r(\cos\theta + i\sin\theta).$$

Here, clearly, $r = |z|$. The angle θ is called the *argument* of the complex number z and is defined only up to a multiple of 2π. From the definition of the complex exponential below it follows that we may write equivalently

$$z = re^{i\theta}.$$

If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ are any two complex numbers, then it is not hard to show that their product has the polar representation

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)},$$

[1] For those who are familiar with fields, we note that we may view the set of complex numbers as the quotient field $\mathbf{R}[x]/(x^2 + 1)$, which is quite natural since we wish it to be the real field $\mathbf{R}$ with a root of the equation $x^2 + 1 = 0$ attached. In this quotient field the equivalence class of x plays the role of i.

[2] Plural, *moduli*.

i.e., that moduli multiply while arguments add. Geometrically, multiplying complex numbers amounts to multiplying lengths and adding angles. Further,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)},$$

i.e., dividing corresponds to dividing lengths and subtracting angles. If m is any positive integer, then, we have

$$z^m = \underbrace{z \cdot z \cdots z}_{m \text{ times}} = r^m e^{im\theta}.$$

If m is a negative integer, we define

$$z^m = \frac{1}{z^{|m|}},$$

and it is simple to show that in this case also

$$z^m = r^m e^{im\theta},$$

i.e., that this formula holds for all nonzero integer exponents m. For nonzero z we define $z^0 = 1$, and this formula then holds for all integer exponents.