APPENDIX II. BASIC DEFINITIONS AND PROPERTIES OF COMPLEX NUMBERS.

1. Basic definitions. A *complex number* is an abstract quantity $z = a + ib$, where a and b are real numbers and i is an abstract quantity which we require to satisfy $i^2 = -1$.[1] We require that these numbers satisfy all of the usual properties of arithmetic; thus if $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we have

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$
$$z_1 \cdot z_2 = a_1a_2 - b_1b_2 + i(a_1b_2 + b_1a_2.$$

We define the *conjugate* of a complex number $z = a + ib$ to be the complex number

$$\overline{z} = a - ib,$$

and note that the product

$$z\overline{z} = a^2 + b^2$$

is always real and nonnegative. By the Pythagorean theorem, $\sqrt{z\overline{z}}$ is the distance from the origin to the point (a, b), and we call this quantity the *modulus*[2] (or, sometimes, the *absolute value* or even the *length*) of the complex number z, and denote it as

$$|z| = \sqrt{z\overline{z}}.$$

The function $|\cdot|$ satisfies all of the usual properties of the absolute value function on real numbers:

$$|z_1z_2| = |z_1||z_2|, \qquad |z_1 + z_2| \le |z_1| + |z_2|.$$

Since geometrically $|\cdot|$ represents distance from the origin, a set of the form

$$\{z||z - z_0| < R\}$$

is a circle of radius R centred at the point corresponding to z_0.

The ratio of two complex numbers can be determined as follows. Let $z_1 = a+ib$, $z_2 = c+id$ be complex numbers with $z_2 \neq 0$; then

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(a+ib)(c-id)}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}.\end{aligned}$$

Let $z = a + ib$ be a complex number, and consider the corresponding point (a, b) in the plane. Let this point have a polar representation (r, θ), where r is the distance to it from the origin and θ is the angle from the positive x axis to a ray from the origin to the point, measured counterclockwise. Then we may write

$$a = r\cos\theta, \qquad b = r\sin\theta,$$

so that we have

$$z = r(\cos\theta + i\sin\theta).$$

Here, clearly, $r = |z|$. The angle θ is called the *argument* of the complex number z and is defined only up to a multiple of 2π. From the definition of the complex exponential below it follows that we may write equivalently

$$z = re^{i\theta}.$$

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[1] For those who are familiar with fields, we note that we may view the set of complex numbers as the quotient field $\mathbf{R}[x]/(x^2 + 1)$, which is quite natural since we wish it to be the real field $\mathbf{R}$ with a root of the equation $x^2 + 1 = 0$ attached. In this quotient field the equivalence class of x plays the role of i.

[2] Plural, *moduli.*

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are any two complex numbers, then it is not hard to show that their product has the polar representation

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)},$$

i.e., that moduli multiply while arguments add. Geometrically, multiplying complex numbers amounts to multiplying lengths and adding angles. Further,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)},$$

i.e., dividing corresponds to dividing lengths and subtracting angles. If m is any positive integer, then, we have

$$z^m = \underbrace{z \cdot z \cdots z}_{m \text{ times}} = r^m e^{im\theta}.$$

If m is a negative integer, we define

$$z^m = \frac{1}{z^{|m|}},$$

and it is simple to show that in this case also

$$z^m = r^m e^{im\theta},$$

i.e., that this formula holds for all nonzero integer exponents m. For nonzero z we define $z^0 = 1$, and this formula then holds for all integer exponents.