Summary:

- We give a basic introduction to the concept of *analytic continuation*, and relate it to our discussion of the logarithm.

102. Analytic continuation. Suppose that we have a function f which is analytic on some region containing a point a. Since f is analytic near a, we may expand it in a Taylor series near a and write

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)(z-a)^k, \tag{1}$$

and the series will converge to f on the largest disk centred at a on which f is analytic. (A good example to keep in mind here might be something simple like $f(z) = 1/z$, expanded around some $a \neq 0$.) Suppose that the series (1) converges on the disk D, necessarily centred at a. Now instead of starting with the function f, suppose that we instead start with the series (1); in other words, suppose that we simply given a series

$$\sum_{k=0}^{\infty} a_k (z-a)^k, \tag{2}$$

which we know converges on the disk D. Let us denote its sum by $f_1(z)$ when $z \in D$ (note that we must have $f_1(z) = f(z)$ where f is analytic). Suppose that D has radius $r > 0$, and let b be a point in D with $|b - a| = r - \epsilon$, where ϵ is small; in other words, b is close to the boundary of D. Now since $b \in D$, the series (2) must converge to an analytic function at b; we may expand thus expand f_1 in a Taylor series about b and write

$$f_1(z) = \sum_{k=0}^{\infty} b_k (z-b)^k. \tag{3}$$

This series will also converge on a disk, say D', which must have radius at least ϵ (since the disk of radius ϵ about b will be contained in the disk D by the triangle inequality). However, D' may have radius bigger than ϵ; in that case, we say that it gives an *extension* of the function f_1 to the disk D'.

Now clearly there is nothing to keep us from continuing on in this fashion, by taking further points in D or in D' and expanding the functions around those points to see whether we get further extensions. A treatment of this is given in Goursat, Chapter IV; sections 83, 84, and 86 in particular are germaine to what we shall do here, but we do not have time to go into all of the details and shall content ourselves with a particular example.

103. A particular example Let us consider the branch of the complex logarithm obtained by taking a cut along the negative real axis and requiring the angle to lie in thet set $(-\pi, \pi)$, and denote the resulting branch by L; in other words, for $\theta \in (-\pi, \pi)$ and $r \in (0, +\infty)$, we define

$$L(re^{i\theta}) = \log r + i\theta.$$

This function is analytic at $z = 1$, and thus has a power series expansion there – which we shall not give as finding it is one of the practice problems on the final review sheet; let us denote it by

$$\sum_{k=0}^{\infty} a_k (z-1)^k. \tag{4}$$

Now this series will converge to L on the largest disk centred at 1 on which L is analytic, which is easily seen to be the disk of radius 1 centred at $z = 1$, since L has a singularity at $z = 0$. Let us call this disk D_1. Since $L(re^{i \cdot 0}) \to -\infty$ as $r \to 0^+$, we see that the series (4) cannot converge on any larger disk.

Now let $b = 1 + i/2$; note that $b \in D_1$, so the series (4) converges at b. Thus we may expand the resulting function about b, obtaining

$$\sum_{k=0}^{\infty} b_k (z-b)^k. \tag{5}$$

Now of course on D_1 the series (4) converges to L, so we are really just expanding the function L around $z = b$ in this case. Thus, once again, the series will converge to L on the largest disk about b which does not include any singularity of L; we see that, as above, this will be the disk D_2 of radius $\sqrt{5/4}$ about b.

Continuing one step further, let us let $c = -1/10 + i/2$; then $|b - c|^2 = 11^2/10^2 = 121/100 < 5/4$, so $c \in D_2$. Thus we may expand the function obtained from the series (5) about $z = c$, obtaining the series

$$\sum_{k=0}^{\infty} c_k (z - c)^k; \tag{6}$$

as before we are actually expanding the function L, so we know that the series will converge to L on any disk about c on which L is analytic. This time, though, there is an extra twist. As above, clearly the largest disk on which the series (6) can converge is the disk, call it D_3, of radius $|c| = \sqrt{\frac{51}{100}}$ about c. However, clearly the point $-1/10$ satisfies $|c - (-1/10)| = 1/2 < \sqrt{\frac{51}{100}}$; in other words, $-1/10 \in D_3$. But $-1/10$ was on the branch cut of L – in other words, the function L *is not defined* at the point $-1/10$!

Let us investigate what is going on here in more detail. Let L_2 denote the branch of Log with a cut along the negative *imaginary* axis and with the angle required to lie in $(-\pi/2, 3\pi/2)$. Then clearly $L = L_2$ on the fourth, first, and second quadrants (i.e., where $\theta \in (-\pi/2, \pi)$); thus the series (4) and (5) are also the Taylor series for L_2 about $z = 1$ and $z = b$, respectively. Similarly, the series (6) is the Taylor series for L_2 about $z = c$, and will converge to L_2 on the largest disk about c on which L_2 is defined. Now geometrically it is clear that this disk is indeed the disk D_3 – in other words, the series (6) will indeed converge everywhere on D_3. It will converge to the function L_2 there. Now, as noted, $L_2 = L$ for $\theta \in (-\pi/2, \pi)$; but a little thought shows that $L_2 - L = 2\pi i$ if $\theta \in (\pi, 3\pi/2)$ – they are distinct branches over that interval.

This means, in particular, that for any $z = re^{i\theta} \in D_3$ with $\theta \in (\pi, 3\pi/2)$, the series in (6) will converge to a value equal to $L(z) + 2\pi i$. Now note that the difference of limits

$$\lim_{\theta \to \pi^-} L(re^{i\theta}) - \lim_{\theta \to -3\pi/2^+} L(re^{i\theta}) = 2\pi i$$

for any r; in other words, the function L has a jump discontinuity equal to $2\pi i$ across the branch cut – it decreases by $2\pi i$ as we cross the branch cut. But this means that if we add $2\pi i$ to L as we cross the branch cut, the result will be continuous (as long as we stay close to the branch cut, of course!) – and this is exactly what the series (6) accomplishes! In other words, given only local information about the branch L, the series was somehow able to pick out the branch L_2 which should follow L across the branch cut in order to have a function which is analytic on both sides of the branch cut.

Now we know of course that there is no branch of the logarithm which is analytic on the entire punctured plane, so the process of extension given above must run into some irresolvable obstacle at some point. Let us see what that is. Suppose that we continue expanding the function L_2 on yet another disk D_4 which is counterclockwisely further around the origin than D_3; we are evidently able to do so. If we then continue, using disks say D_5, D_6, etc., at some point we shall run into the same situation with L_2 which we had with L: the series will converge across the branch cut, but to another branch, call it L_3, on the other side of the cut, i.e., for $\theta > 3\pi/2$. As before, we will have $L_3 - L_2 = 2\pi i$ for points just across the branch cut; but $L_2 = L$ there, so $L_3 - L = 2\pi i$. Now we may take L_3 to be the branch with a cut along the positive imaginary axis and the angle θ required to lie in $(3\pi/2, 7\pi/2)$. Continuing to expand as before, we will eventually arrive back at the point $z = 1$. However, at that point we will be expanding the function L_3 instead of L, and that means that the series will converge to $L_3(1) = L(1) + 2\pi i = 2\pi i$, not $L(1) = 0$!

Note the close analogy of this procedure to what we discussed much earlier in the course about how integrating $1/z$ around the origin to get $\operatorname{Log} z$ will increase the value by $2\pi i$ – exactly the value determined here. The procedure here is however far more general, and in particular could be applied to the root functions, giving the different branches in cyclic succession as we continue expanding around the origin. (In other words, if we start with one branch of – say – the cube root function, say that which gives $1^{1/3} = 1$, and then expand it in Taylor series which circle the origin as here, then when we come back to the point 1 again we will have instead $e^{2\pi i/3}$; if we circle again, we will have $e^{4\pi i/3}$; and if we circle once more – making three times in total – we will come back to the original value, 1. With the logarithm, we would continue adding $2\pi i$ each time, which means we will never return to the original value no matter how many times we circle.) Similarly, we could apply this to the function in the last problem on the term test: in particular, the last part shows that if we were to start with the function on one side of the branch cut, and then expand it in series on disks which wrapped around the branch cut to the other side, by the time we came back to the original point there would be a difference of $-\pi/(2\sqrt{2})$.

Those who have seen – or will yet see – covering spaces should note the similarity here to the construction of the universal cover of the circle, or the punctured plane (which is homotopically equivalent, in fact there is an obvious deformation retract of the punctured plane onto the circle): we start at a certain point and then start taking curves, reducing by homotopy; since a closed loop once around the origin is not homotopic to a point, the endpoint of this curve is taken to represent a distinct point from its initial point. Similarly, a closed loop *twice* about the origin is not homotopic to either of these paths, meaning that *its* endpoint is yet another distinct point, and so on. Another way of putting all of this together is that, should we define the logarithm on the universal cover of the punctured plane instead of the punctured plane itself, it would become a single-valued analytic function. Taking a branch would then correspond to restricting the domain to some piece of this universal cover for which the covering map is a homeomorphism onto a cut plane. (What we just noted about root functions shows that something similar is true for them, but instead of needing to use the universal cover of the punctured plane, we need to use, for the nth-root function, the n-sheeted cover.) Similarly, functions with more complicated branch points – such as the function on the term test, which has four branch points; or the arctangent function, which has two – can be defined on covers of the multiply-punctured plane. The fact that if we traverse a loop around all four points we come back to the same value means however that we do not get the universal cover of the quadruply-punctured plane in this case, but rather some other set, an elucidation of which is however beyond the knowledge of the present author, who will therefore retire before he says anything more wrong than he already has.