Summary:
- We give a proof of L'Hôpital's rule for analytic functions.
- We then show how another class of boundary conditions for Laplace's equation transforms under analytic maps, and give an example.
- [For the material on analytic continuation, please see the pre-class notes.]

39. L'Hôpital's rule for analytic functions. We prove the following version of L'Hôpital's rule for analytic functions. Suppose that f and g are analytic near a point a, and that both f and g have a zero of order m at a. Then

$$\lim_{z\to a}\frac{f(z)}{g(z)}=\lim_{z\to a}\frac{f^{(m)}(z)}{g^{(m)}(z)}. \tag{1}$$

(In the terminology usually used to discuss L'Hôpital's rule in elementary calculus courses, this is a limit of type 0/0.)

To see this, note that since f and g have zeroes of order m at a, there are functions $\phi(z)$ and $\gamma(z)$ which are both analytic and nonzero near a and satisfy

$$f(z)=(z-a)^m\phi(z),\qquad g(z)=(z-a)^m\gamma(z).$$

Moreover, we claim that $\phi(a)=f^{(m)}(a)/m!$ and $\gamma(a)=g^{(m)}(a)/m!$. The proof of these two equations is the same so we show only the first one. There are two distinct ways of doing this. The one we used in lecture involved Taylor series and is as follows. Suppose that the Taylor series for ϕ at a is

$$\sum_{k=0}^{\infty}a_k(z-a)^k,$$

where we know that $a_0=\phi(a)\neq 0$. Then the Taylor series for f at a is

$$\sum_{k=0}^{\infty}a_k(z-a)^{k+m}.$$

But this series must equal

$$\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(z-a)^n;$$

equating coefficients of like powers, we see that

$$a_0=\frac{1}{m!}f^{(m)}(a),$$

as claimed.

The second method is much quicker and involves the Cauchy integral formula: if C is a sufficiently small circle around a, then we have

$$\begin{aligned}\phi(a)&=\frac{1}{2\pi i}\int_C\frac{\phi(z')}{z'-a}\,dz'\\&=\frac{1}{2\pi i}\int_C\frac{f(z')/(z'-a)^m}{z'-a}\,dz'=\frac{1}{2\pi i}\int_C\frac{f(z')}{(z'-a)^{m+1}}\,dz'\\&=\frac{1}{m!}f^{(m)}(a),\end{aligned}$$

by the Cauchy integral formula for derivatives.

Given this, the proof of (1) is easy:

$$\lim_{z\to a}\frac{f(z)}{g(z)}=\lim_{z\to a}\frac{(z-a)^m\phi(z)}{(z-a)^m\gamma(z)}=\lim_{z\to a}\frac{\phi(z)}{\gamma(z)}=\frac{\phi(a)}{\gamma(a)}=\frac{f^{(m)}(a)}{g^{(m)}(a)},$$

since the factors of $m!$ will cancel.

We can also use the inner workings of the above proof to evaluate limits, which is often much easier than applying the result itself, as the following example shows.

EXAMPLE. Let $n \in \mathbf{Z}$ be positive. Find

$$\lim_{x \to 0} \frac{(\sin x)^{2n}}{(1 - \cos x)^n}.$$

Let us consider the corresponding complex limit, i.e., replace the real number x in the limit above by a complex number z, since if the resulting limit exists then certainly the original limit does as well. Now note that $\sin z$ has a zero of order 1 at 0, so that we may write

$$\sin z = z\phi(z)$$

for some function ϕ which will be analytic and nonzero at $z = 0$ (and hence everywhere in this case, though that is not important). In fact, of course, the function ϕ will simply be the function

$$\begin{cases} \sin z/z, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

which we have seen many times already. We have moreover that $\phi(0) = \frac{d}{dz} \sin z\big|_{z=0} = 1$. Further, since

$$\cos z = 1 - \frac{1}{2}z^2 + \cdots, \tag{2}$$

we see that $1 - \cos z$ has a zero of order 2 at 0, so we may write

$$1 - \cos z = z^2\gamma(z),$$

where by inspection of the series (2) we have $\gamma(0) = \frac{1}{2}$. (This could also be obtained by differentiating as in the proof above, of course; in that case the factor of $1/2$ comes from the $1/m!$.) Thus we may write

$$\begin{aligned} \lim_{z \to 0} \frac{(\sin z)^{2n}}{(1 - \cos z)^n} &= \lim_{z \to 0} \frac{z^{2n}[\phi(z)]^{2n}}{z^{2n}[\gamma(z)]^n} \\ &= \lim_{z \to 0} \frac{[\phi(z)]^{2n}}{[\gamma(z)]^n} = \frac{[\phi(0)]^{2n}}{[\gamma(0)]^n} = 2^n. \end{aligned}$$

To apply L'Hôpital's rule to this directly would have required us to differentiate n times, which would be extremely messy. This demonstrates the utility of power series manipulations and other concepts (such as the order of a zero) which we have studied in this course in elucidating the behaviour of analytic functions.

40. Transformation of Neumann boundary conditions by analytic maps. So far we have considered only problems involving so-called *Dirichlet* boundary conditions, i.e.,

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = h. \tag{3}$$

This means, of course, that we seek a function u which satisfies Laplace's equation inside the region D and is equal to the function h on the boundary of D; note that the Laplacian of u need not be defined on the boundary of D. Another type of boundary condition we could give is that the normal deriviative of u on the boundary be equal to some given function. In other words, let $\mathbf{n}$ denote the unit outward normal to the boundary curve ∂D (this is defined in an analogous way to how one defines the outward unit normal to a region in three-dimensional space when one discusses the divergence theorem in multivariable calculus); then, writing

$$\frac{\partial u}{\partial n} = \mathbf{n} \cdot \nabla u$$

(i.e., we *define* $\partial u/\partial n$ to be the quantity on the right-hand side), we may consider the problem

$$\Delta u = 0 \text{ on } D, \qquad \frac{\partial u}{\partial n}\bigg|_{\partial D} = h, \tag{4}$$

where again h is some function on the boundary ∂D. Note first of all that this problem will not have a unique solution on D, since if u is any solution then clearly so also is $u+C$, where C is any real number; thus in order to get a unique solution we need to specify some other piece of information about u, for example, its value at some point. We shall assume that given the value of u at any point, (4) has a unique solution. We shall also restrict attention to the case $h=0$.

Let us see whether our method of transforming the problem with an analytic function is applicable to problem (4). Thus consider a region E and an analytic function $f : E \to D$ satisfying the conditions given in §38 of these lecture notes, and consider problem (4) with $h = 0$. We claim that if $v = u \circ f$, then v satisfies the problem

$$\Delta v = 0 \text{ on } E, \qquad \left.\frac{\partial v}{\partial N}\right|_{\partial E} = 0, \tag{5}$$

where $\partial/\partial N$ denotes the derivative of v in the direction normal to E. Let us see why this is so. That v satisfies $\Delta v = 0$ has already been shown; thus we need only consider the boundary condition. Let $a \in E$. Let $\mathbf{N}$ denote the outward unit normal to E at a; then

$$\left.\frac{\partial v}{\partial N}\right|_{a} = \mathbf{N} \cdot \nabla v(a) = \mathbf{N} \cdot \nabla (u \circ f)(a).$$

Now let $b = f(a)$ and consider $\nabla u(b)$. If $\nabla u(b) = 0$, then it is not hard to show that $\nabla(u \circ f)(a) = 0$ (using the chain rule); thus in this particular case $\partial v/\partial N|_a$ will be zero, as claimed. Thus now suppose that $\nabla u(b) \neq 0$. Since

$$\left.\frac{\partial u}{\partial n}\right|_{b} = \mathbf{n} \cdot \nabla u(b) = 0,$$

where $\mathbf{n}$ denotes the outward unit normal to D at $b = f(a)$, we see that $\nabla u(b)$ must be perpendicular to $\mathbf{n}$. But now $\nabla u(b)$ must also be perpendicular to the level curve, call it C, of u which passes through b, i.e., to the set of points (x, y) satisfying $u(x, y) = u(b)$; since we are in the plane this implies that this level curve must be tangent to $\mathbf{n}$ and hence perpendicular to the boundary curve ∂D. But since f^{-1} is assumed to be analytic and therefore conformal (its derivative will be nonzero since it is invertible), and maps ∂D to ∂E, this means that the curve C' to which f^{-1} maps C must be perpendicular to the boundary ∂E at the point a. But this curve is just the level curve of v through a, for (x, y) satisfies $v(x, y) = v(a)$ if and only if

$$u[f(x, y)] = u[f(a)] = u(b),$$

i.e., if and only if $f(x+iy) = x'+iy'$ for some point (x', y') in C, which is equivalent to $x+iy = f^{-1}(x'+iy')$, i.e., that (x, y) lie on C'. Thus the level curve of v through a must be perpendicular to the boundary ∂E, which means as before that the gradient of v at a must be perpendicular to the normal vector $\mathbf{N}$, and hence

$$\left.\frac{\partial v}{\partial N}\right|_{a} = \mathbf{N} \cdot \nabla v(a) = 0,$$

so that v satisfies the boundary condition in (5), as claimed.

Before going into the example, recall that the map $z \mapsto \sin z$ was described at the end of §38 in the previous set of lecture notes.

EXAMPLE. Let $U = \{(x, y)|y > 0\}$ denote the upper half-plane. Solve the following problem:

$$\Delta u = 0 \text{ on } U, \qquad u|_{\partial U} = \begin{cases} 0, & x \in (-\infty, -1) \\ 1 & x \in (1, \infty) \end{cases}, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial U} = 0 \text{ if } x \in [-1, 1]; \tag{6}$$

in other words, we require u to be 0 on the real axis except on the interval $[-1, 1]$, and we require $\partial u/\partial n$ to be zero on this interval. Now geometrically it is clear that $\partial u/\partial n = -\partial u/\partial y$, so that this is equivalent to requiring $\partial u/\partial y$ to vanish on this interval.

Now of the conformal maps listed at the end of §38, the only one which maps onto the upper half-plane is $z \mapsto \sin z$, where we take z to lie in the rectangle

$$R = \{(x, y)|x \in (-\pi/2, \pi/2),\ y > 0\}.$$

Thus we consider transforming the problem (6) using this map. Let $v(z) = u[\sin(z)]$; then v must satisfy the problem

$$\nabla v = 0 \text{ on } R, \qquad v(-\pi/2, y) = 0, \qquad v(\pi/2, y) = 1, \qquad \left.\frac{\partial v}{\partial N}\right|_{(x,0)} = 0,\ x \in (-\pi/2, \pi/2), \tag{7}$$

where here $\partial/\partial N$ denotes the outward normal derivative of v; but as before this is just $-\partial/\partial y$, so that this last condition gives simply

$$\left.\frac{\partial v}{\partial y}\right|_{(x,0)} = 0,\ x \in (-\pi/2, \pi/2).$$

Thus we must now find a function v which satisfies (7). Since the only real class of solutions we have on rectangles are the linear ones, let us see whether this problem has a linear solution; in other words, let us see whether we can find numbers a, b, and c such that

$$v(x, y) = a + bx + cy$$

(which will automatically satisfy $\Delta u = 0$) satisfies the boundary conditions. The conditions on $x = \pm\pi/2$ give

$$0 = v(-\pi/2, y) = a - b\pi/2 + cy, \qquad 1 = v(\pi/2, y) = a + b\pi/2 + cy,$$

whence we see immediately that $c = 0$ (as otherwise the right-hand sides of these expressions would depend on y while the left-hand sides clearly do not) and are thus left with the system

$$\begin{aligned} a - \frac{\pi}{2}b &= 0 \\ a + \frac{\pi}{2}b &= 1. \end{aligned}$$

Adding these equations gives $a = 1/2$, while subtracting the first from the second gives $b = 1/\pi$; thus the function

$$v = \frac{1}{2} + \frac{1}{\pi}x$$

will indeed satisfy the boundary conditions on $x = \pm\pi/2$, a fact which can readily be verified by direct substitution. Note that this solution satisfies also $\partial v/\partial y = 0$ everywhere, and hence in particular on the segment $\{(x, 0) \mid x \in (-1, 1)\}$, which is the final boundary condition in (7). Thus v is the solution to (7).

This means that the solution to the original problem will be $u = v \circ \arcsin z$. Now if we write a point in the plane in complex notation, then we have $v(x + iy) = \frac{1}{2} + \frac{1}{\pi}x = \frac{1}{2} + \frac{1}{\pi}\mathrm{Re}\,(x + iy)$; thus we need to determine $\mathrm{Re} \arcsin z$, where z is in the upper half-plane and the branch of arcsin taken is that which maps into the rectangle R. We could determine this using the formula for arcsin which we found previously, but it seems easier to proceed in a different way. Let $z = a + ib$ be in the upper half-plane, and let $w = x + iy \in R$ satisfy $\sin w = z$; in other words,

$$\sin w = \sin x \cosh y + i \cos x \sinh y = a + ib,$$

and we note that $b > 0$, which implies that $x \in (-\pi/2, \pi/2)$ and $y > 0$. We wish to find $\mathrm{Re}\, w = x$. Now as we just saw, $b > 0$ implies $\cos x > 0$. Further, $\sin x = 0$ exactly when $a = 0$, and in this case we must have $x = 0$. Thus for us $\mathrm{Re}\, arcsin 0 = 0$, and we may assume that $a \neq 0$. Now let $\alpha = \sin^2 x$. Then $\cos^2 x = 1 - \alpha$, so that we have

$$\frac{a^2}{\alpha} + \frac{b^2}{\alpha - 1} = \frac{a^2}{\sin^2 x} - \frac{b^2}{\cos^2 x} = \cosh^2 y - \sinh^2 y = 1;$$

if we multiply through by $\alpha(\alpha - 1)$, this gives

$$\begin{aligned} \alpha^2 - \alpha &= a^2(\alpha - 1) + \alpha b^2 = \alpha(a^2 + b^2) - a^2 \\ \alpha^2 - (1 + a^2 + b^2)\alpha + a^2 &= 0 \end{aligned}$$

whence applying the quadratic formula, we obtain

$$\alpha = \frac{1}{2}\left[1 + a^2 + b^2 \pm \sqrt{(1 + a^2 + b^2)^2 - 4a^2}\right]. \tag{8}$$

That the quantity inside the square root is always positive can be seen as follows:

$$\begin{aligned}(1 + a^2 + b^2)^2 - 4a^2 &= 1 + 2a^2 + 2b^2 + a^4 + 2a^2b^2 + b^4 - 4a^2 \\ &= b^2(b^2 + 2) + a^4 - 2a^2 + 1 + 2a^2b^2 = b^2(b^2 + 2) + 2a^2b^2 + (a^2 - 1)^2,\end{aligned}$$

which is a sum of positive (since $b \neq 0$) and nonnegative quantities and therefore positive. Now note that both roots above are nonnegative, since $\sqrt{(1 + a^2 + b^2)^2 - 4a^2} \leq 1 + a^2 + b^2$. We claim that the + root in (8), which we denote by α_+, must be greater than 1. To see this, note that since $b > 0$

$$4 - 4(1 + b^2) = -4b^2 < 0;$$

thus

$$(1+b^2+a^2)^2 - 4a^2 > (1+b^2+a^2)^2 - 4(a^2+b^2) = (1+b^2+a^2)^2 - 4(1+a^2+b^2) + 4 = [(1+b^2+a^2) - 2]^2,$$

so

$$\sqrt{(1 + b^2 + a^2)^2 - 4a^2} > \left|1 + b^2 + a^2 - 2\right|.$$

Now if $1 + b^2 + a^2 > 2$, then clearly $\alpha_+ > 1$; while if $1 + b^2 + a^2 \leq 2$, then by the above inequality we have

$$\sqrt{(1 + b^2 + a^2)^2 - 4a^2} > 2 - (1 + b^2 + a^2), \qquad (1 + b^2 + a^2) + \sqrt{(1 + b^2 + a^2)^2 - 4a^2} > 2,$$

and again $\alpha_+ > 1$. Thus in any event we must have $\alpha_+ > 1$, as desired. Since we require $\alpha = \sin^2 x$ for x real, we are only interested in values of α that lie in $[0, 1]$, and we therefore reject the value α_+ and take $\alpha = \frac{1}{2}\left[1 + a^2 + b^2 - \sqrt{(1 + a^2 + b^2)^2 - 4a^2}\right]$. We claim that $\alpha < 1$. This can be seen as follows:

$$\begin{aligned}2\alpha = 1 + a^2 + b^2 - \sqrt{(1 + a^2 + b^2)^2 - 4a^2} &= \frac{(1 + a^2 + b^2)^2 - [(1 + a^2 + b^2)^2 - 4a^2]}{1 + a^2 + b^2 + \sqrt{(1 + a^2 + b^2)^2 - 4a^2}} \\ &= \frac{4a^2}{1 + a^2 + b^2 + \sqrt{b^2(b^2 + 2) + 2a^2b^2 + (a^2 - 1)^2}} < \frac{4a^2}{1 + a^2 + b^2 + a^2 - 1} < \frac{4a^2}{2a^2} = 2,\end{aligned}$$

giving $\alpha < 1$, as desired. (Here we have used $b > 0$ to conclude that $\sqrt{b^2(b^2 + 2) + 2a^2b^2 + (a^2 - 1)^2} > \sqrt{(a^2 - 1)^2} = |a^2 - 1| \geq a^2 - 1$.) This gives, finally, then,

$$x = \operatorname{Re} \arcsin z = \pm \arcsin \sqrt{\frac{1}{2}\left[1 + a^2 + b^2 - \sqrt{(1 + a^2 + b^2)^2 - 4a^2}\right]},$$

where here arcsin denotes the ordinary inverse sine of a number in the interval $[0, 1)$, lying in $[0, \pi/2)$. We will take the + sign for $a \geq 0$ and the − for $a \leq 0$; $a = 0$ clearly gives $x = 0$, so in this case it does not matter which sign we take. Our solution is then finally

$$u(x, y) = \frac{1}{2} + \frac{1}{\pi} \cdot \begin{cases} \arcsin\sqrt{\frac{1}{2}\left[1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2}\right]}, & x \geq 0 \\ -\arcsin\sqrt{\frac{1}{2}\left[1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2}\right]}, & x \leq 0 \end{cases}. \tag{9}$$

It is instructive to see how this solution satisfies the boundary conditions. Suppose $y = 0$; then the formula above gives[1]

$$1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2} = 1 + x^2 - \sqrt{(1 - x^2)^2} = 1 + x^2 - |1 - x^2|.$$

[1] Technically, of course, we should be considering rather the *limit* as $y \to 0^+$. The formula above will clearly be continuous at $y = 0$, however, if it is defined at $y = 0$; and thus it suffices to show that it is defined at $y = 0$, which is afforded by our following calculation.

Now if $x \in [-1,1]$, this gives $1 + x^2 - (1 - x^2) = 2x^2$, while if $x \in (-\infty,-1) \cup (1,+\infty)$ it gives instead $1 + x^2 - (x^2 - 1) = 2$. Thus

$$\arcsin \sqrt{\frac{1}{2}\left[1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2}\right]} = \begin{cases} \arcsin |x|, & x \in [-1,1] \\ \frac{\pi}{2}, & x \in (-\infty,-1) \cup (\infty,1) \end{cases},$$

where the second line follows from $\arcsin 1 = \pi/2$. If we substitute this back into (9), we thus obtain (using the fact that arcsin is odd)

$$\begin{aligned} u(x,0) &= \frac{1}{2} + \frac{1}{\pi}\begin{cases} -\frac{\pi}{2}, & x \in (-\infty,-1) \\ \arcsin x, & x \in [-1,1] \\ \frac{\pi}{2}, & x \in (1,+\infty) \end{cases} \\ &= \begin{cases} 0, & x \in (-\infty,-1) \\ \frac{1}{2} + \frac{1}{\pi}\arcsin x, & x \in [-1,1] \\ 1, & x \in (1,+\infty) \end{cases}, \end{aligned}$$

which clearly satisfies the boundary conditions on $(-\infty,-1) \cup (\infty,1)$. That the remaining condition on $(-1,1)$ holds, we leave to the intrepid reader; it follows fairly simply from the observation that

$$\frac{\partial}{\partial y}\left[1 + x^2 + y^2 - \sqrt{(1 + x^2 + y^2)^2 - 4x^2}\right] = 2y - 2y(1 + x^2 + y^2)[(1 + x^2 + y^2)^2 - 4x^2]^{-1/2},$$

and the observation that this quantity vanishes as $y \to 0^+$ for all $x \in (-1,1)$. (This is clear, since the expression inside the exponent is positive for all such x even when $y = 0$, by the foregoing.)