Summary:

- We give a more careful treatment of the method of solving Laplace's equation using linear solutions.

103. Linear solutions to boundary value problems for Laplace's equation. Basically, when we are solving a boundary-value problem, we are trying to find, by whatever method, a function u which satisfies two conditions: (i) it must satisfy Laplace's equation, i.e., its Laplacian must vanish; (ii) its value on the boundary of the region must equal the specified function. 'By whatever method' means that we do not care how we obtained the function, only that it satisfies these two conditions; in other words, the method does not need to be constructive or computational in any way. (This is all right since there are general theorems which guarantee that the solutions to problems like this are unique, at least when the boundary data is sufficiently nice.) Obviously, then, one way of 'finding' the solution would be to try one function after another until (hopefully) the correct one is found. Since there are infinitely many different functions we might need to try, though, that isn't a very practical idea. On the other hand, most functions one could think of writing down will certainly not satisfy condition (i); for example, $\sin x$ certainly doesn't, nor does $\sin x \sin y$, etc.. So maybe a good starting point would be to try to find a collection of functions (ideally all functions, though in practice that is again too many) which satisfy Laplace's equation, and then see if maybe we can somehow find one of them which also has the correct values on the boundary. Now one especially simple class of functions which satisfy Laplace's equation are the linear functions, $g(x, y) = a + bx + cy$: this is because we can calculate:

$$\frac{\partial g}{\partial x} = a, \qquad \frac{\partial g}{\partial y} = b,$$
$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x} a = 0, \qquad \frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y} b = 0,$$

so

$$\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0.$$

Now that we know that these functions all satisfy Laplace's equation, if we are trying to solve a boundary-value problem we only need to find *numbers* a, b, and c such that the boundary conditions are satisfied. In other words, we are going to substitute the linear solution $a + bx + cy$ into the boundary conditions in (ii) and try to solve for the numbers a, b, and c. (For anything other than very special boundary conditions, of course, this will not be possible, because the linear solutions are too special; but for the problems on this assignment this is possible at some point.) Generally we solve for the numbers a, b and c either 'by inspection' or by substituting in values for x and y to obtain a system of equations that they must satisfy, which we then try to solve.

Consider the following trivial examples on the unit square $D = \{(x, y) \,|\, x, y \in [0, 1]\}$:

EXAMPLE 1. Solve the following problem:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = 1.$$

If we try a linear solution in this case, then we would say, let us see whether a solution of the form $u = a + bx + cy$ can solve this problem. It clearly solves $\Delta u = 0$; now the condition $u|_{\partial D} = 1$ means $a + bx + cy = 1$ whenever $(x, y) \in \partial D$. Thus, for example, if we let $x = 1$, $y = 0$ (which is clearly a point in ∂D), we get $a + b = 1$; if we let $x = 1$, $y = 1$, we get $a + b + c = 1$, which means $c = 0$; if we let $x = 0$, $y = 0$, we get $a = 1$, which also means that $b = 0$ since $a + b = 1$. Thus the only linear function which could possibly satisfy the boundary conditions is $u = 1$. But this function clearly does actually satisfy the boundary condition on all of ∂D, and since it satisfies Laplace's equation (as it had to since it was a particular example of the class of linear functions, each one of which is a solution to Laplace's equation), it must be the desired solution.

EXAMPLE 2. Solve the following problem:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = x.$$

Let us again try a linear solution: thus we wish to determine whether there are constants a, b, and c such that $u = a + bx + cy$ (which must satisfy Laplace's equation) satisfies the boundary condition. In this case things get a bit more interesting. Suppose that $y = 0$; then we have, for all $x \in [0, 1]$, that $a + bx = x$. If we set $x = 0$, this gives $a = 0$, so $bx = x$; if we set $x = 1$ (or, for that matter, if we let x be any nonzero number), this gives $b = 1$. So far, then, we know that we must have $u = x + cy$. Now if $x = 0$ and $y \neq 0$, say $y = 1$, then we have by the boundary condition that $cy = c = 0$. Thus the only linear solution that could possibly satisfy the boundary conditions is $u = x$. Now this does actually clearly satisfy the boundary conditions; and since it also satisfies Laplace's equation, it must be the desired solution.

Note however that the solution cannot always be read off from the boundary conditions as in the two examples above. For example, the boundary condition in example 2 could have been expressed as follows: $u|_{\partial D} = 0$, $x = 0$, 1, $x = 1$, x, $y = 0$ or $y = 1$, which we might not immediately recognise – but the method above would give $u = x$ as the only solution regardless. For a more involved example, consider the following problem:

Example 3. Solve the following problem:

$$\Delta u = 0 \text{ on } D, \qquad u|_{\partial D} = \begin{cases} x, & y = 0 \\ x + 1, & y = 1 \\ y, & x = 0 \\ y + 1, & x = 1 \end{cases}$$

Let us see whether we can find a linear solution; thus suppose that $u = a + bx + cy$. The first of the boundary conditions gives $a + bx = x$ for $x \in [0, 1]$; if $x = 0$ this gives $a = 0$, while if $x = 1$ this gives $b = 1$. Thus we already know that if there is such a solution, it must be of the form $u = x + cy$. Now consider the third of the boundary conditions ($u = y$ when $x = 0$); this gives $cy = y$, which, setting $y = 1$, gives $c = 1$. Thus the only possible solution would be $u = x + y$. But now we need to show that this does indeed satisfy the other two boundary conditions. At $y = 1$ this expression gives $u(x, 1) = x + 1$, which is the correct expression; and at $x = 1$ it gives $u(1, y) = 1 + y = y + 1$, which is again the correct expression. Thus $u = x + y$ satisfies the boundary conditions, and since it satisfies Laplace's equation, it must be the desired solution.

These are of course rather simple examples, but they demonstrate the technique. More complicated examples were given in the lecture notes (see August 11, Section 38, pp. 3 – 4, August 13, Section 40, p. 4); it may be helpful to restudy these in the light of the above explanations.

The same technique is applicable to the problems on the last homework assignment: we posit that u takes a certain form, and then determine the coefficients by matching that form to the given boundary conditions.