Summary:
- We review the method of using analytic maps to perform a 'change of variables' in boundary-value problems involving Laplace's equation.
- We give an example, and then study the properties of a few special maps.

(Cf. Goursat, §§22, 24.)

38. Transformations of solutions to Laplace's equation. Last week we discussed the Poisson kernel and its relation to the Cauchy integral theorem: we saw that, just as the Cauchy integral formula gives the value of an analytic function everywhere inside a simple closed curve as an integral involving only the values of the function on the *boundary* of the region (i.e., the simple closed curve), so too the Poisson kernel allows us to write the values of a harmonic function inside a simple closed curve as an integral involving only the values of the function on the boundary (i.e., again, the curve).[1] Today we shall see another method of solving Laplace's equation given information about the function on the boundary, using the fact that analytic functions essentially take harmonic functions to harmonic functions and thus allow us to perform a 'change of variables' of sorts in Laplace's equation to replace a (potentially) hard problem with a (hopefully) easier one.

Before beginning this, we note one small point. When we speak of harmonic functions, we always mean real-valued functions of two real variables. On the other hand, when we speak of analytic functions on the plane we mean complex-valued functions of a complex variable. To avoid tiresome and unimportant notational issues, we shall agree that if u is any real-valued function of two real variables, and $z \in \mathbf{C}$ is any complex number, then the notation $u(z)$ (which is technically undefined) shall mean $u(\operatorname{Re} z, \operatorname{Im} z)$; in other words, we write as a shorthand

$$u(x + iy) = u(x, y).$$

With this out of the way, we have the following result, which we saw some version of back towards the start of the course: suppose that D, $E \subset \mathbf{C}$ are two regions (e.g., interiors of closed curves), that $u : D \to \mathbf{R}$ is a harmonic function on D, and that $f : E \to D$ is an analytic map with $f(E) \cap \partial D = \emptyset$.[2] Then the

[1] It is worth noting however that there is a very basic difference between these two formulas which we have so far glossed over: we have noted (though not proved) that the function u defined using the Poisson kernel approaches the function h in the limit as its argument (x, y) goes radially to a point on C. The same is *not* in general true for the Cauchy integral formula, for a very basic reason. As we noted earlier, the method by which we proved the Cauchy integral formula for derivatives allows us to prove also that any function defined as an integral of the form

$$F(z) = \int_C \frac{h(z')}{z' - z}\, dz', \tag{1}$$

where C is some simply closed curve, z is any point inside C, and h is a continuous function on C, must also have a derivative everywhere inside C, and hence must be analytic. But *this analytic function need not agree with h on the boundary curve C,* for the following reason: suppose that h were real-valued; then if $F(z) = h(z)$ for $z \in C$, the analytic function F would be real-valued on C, and hence its imaginary part would be zero everywhere on C. But its imaginary part must be harmonic, and any harmonic function which vanishes on a simple closed curve must vanish everywhere inside that curve; hence F must be real-valued inside C as well as on C, and must therefore be constant. But if h is not constant then this contradicts $F(z) = h(z)$ on C. Another way of looking at this is that the function F includes two nonzero harmonic functions, so somehow the integral (1) is actually giving us *two* harmonic functions, which clearly contain more information on the boundary than is included in the function h alone. (I should point out that in general there is no reason to believe that even the real part of F agrees with h on the boundary curve C; and here it is worth recalling that when we derived the Poisson kernel we had to add in an extra term by hand involving the point z^*.) This is a bit of a side comment, but worth keeping in mind to avoid making mistakes.

[2] In the applications we shall be mostly interested in cases where the boundary curves of D and E are simple (i.e., where D and E are simply connected), where $f(\partial E) = \partial D$, and where f^{-1} is also analytic. But nothing in this present result relies on any of these conditions, so we state it in greater generality.

function
$$v = u \circ f : E \to \mathbf{R}$$
is also harmonic on E.

To see this, let $a \in E$; then $f(a) \in D$, so that u must be harmonic at $f(a)$. Now we can find an $\epsilon > 0$ so that the disk of radius ϵ centred at $f(a)$ is still contained in D [we do not care whether it is contained in $f(E)$; that is actually totally irrelevant]; let us denote this disk by U. Thus u is actually harmonic on U, and there must then be a conjugate harmonic function on U, call it $\tilde{u}$, so that $u + i\tilde{u}$ is analytic on U. (See Goursat, §3, p. 10; or §9 in these lecture notes.) Let us denote this function by g. Then $g \circ f$ must be analytic at a, by the chain rule; but since the real and imaginary parts of all analytic functions are harmonic, its real part $u \circ f = v$ must be harmonic on E, as desired. (The restricting to a disk is only to make sure that the function $\tilde{u}$ is well-defined – as we have seen before (e.g., in the second quiz), on a non-simply connected region a conjugate harmonic function may become multiple-valued.)

For the applications, we need to restrict the regions E and D and the function f, and for this we need a bit more notation. We shall require E and D to be simply connected; equivalently, we require their boundary curves ∂D and ∂E to be simple closed curves. We shall also require E and D to represent only the interior of their boundary curves, i.e., that $E \cap \partial E = D \cap \partial D = \emptyset$; we set $\overline{E} = E \cup \partial E$, $\overline{D} = D \cup \partial D$.[3] We assume that $f^{-1} : D \to E$ (exists and) is analytic, and that f and f^{-1} extend to continuous functions mapping $\overline{E}$ to $\overline{D}$ and $\overline{D}$ to $\overline{E}$, respectively; at the risk of being extremely confusing, we shall denote these extensions by f and f^{-1} as well.

To put all of this more simply, we assume that f is invertible with analytic inverse, and that f and its inverse extend to continuous functions on the boundary of E and D, respectively.

Now suppose that u satisfies
$$\Delta u = 0 \,\text{on}\, D, \qquad u|_{\partial D} = h, \tag{2}$$
where h is some piecewise-continuous function on ∂D. We call any problem like (2) a *boundary-value problem,* since we have conditions on u on the boundary of the region. Now let E and $f : E \to D$ satisfy the conditions just described, and set $v = u \circ f$. Then v is harmonic on E, while if $a \in \partial E$ we must have $f(a) \in \partial D$, so that
$$v(a) = (u \circ f)(a) = u[f(a)] = h[f(a)]$$
by the boundary condition in (2). Thus v satisfies the boundary-value problem
$$\Delta v = 0 \,\text{on}\, E, \qquad v|_{\partial E} = h \circ f. \tag{3}$$
Now it may happen that we can find E and f such that problem (3) is simpler than (2); in fact, in the examples we shall see the answer to (3) can be guessed at quite easily. Suppose thus that we can find some function v satisfying (3). Then running the above logic backwards, we see that the function $\tilde{u} = v \circ f^{-1}$ must satisfy the boundary-value problem
$$\Delta \tilde{u} = 0 \,\text{on}\, D, \qquad \tilde{u}|_{\partial D} = h;$$
in other words, $\tilde{u}$ is a solution to our original boundary value problem (2).

Let us see an example.

Example. Solve the following boundary problem on the region D given in polar coordinates as $D = \{(r, \theta) \,|\, r \in (0, 1),\, \theta \in (-\pi/4, \pi/4)\}$:
$$\Delta u = 0 \,\text{on}\, D, \qquad u|_{\partial D} = \begin{cases} \sin 2\theta, & r = 1 \\ r^2, & \theta = \pi/4 \\ -r^2, & \theta = -\pi/4 \end{cases} \tag{4}$$

The factor of 2 in the sine, the square in $\pm r^2$, and the shape of the region suggest that perhaps f ought to have something to do with a square. Now $z \mapsto z^2$ increases the angle by a factor of 2; thus if we were to

[3] For those who know something of topology, we note that $\overline{E}$ and $\overline{D}$, as the notation suggests, are just the closure of E and D in the usual topology of $\mathbf{R}^2$ in this case.

take $f(z) = z^2$, the region E would be even narrower than the region D, and the problem would probably not be any simpler. If we take $f(z) = z^{1/2}$, though (assuming we can take an appropriate branch!), then we see that the region E would be a half-disk. Thus let us let $f(z) = z^{1/2}$, where we take a branch cut along the negative real axis and require $\theta \in (-\pi, \pi)$. Then if we set

$$E = \{(r, \theta) \,|\, r \in (0, 1),\, \theta \in (-\pi/2, \pi/2)\},$$

we see that $f : E \to D$ is one-to-one and onto, and its inverse is given by $z \mapsto z^2$, which is analytic on D. Further, we may extend f to a continuous function on $\overline{E}$ by setting $f(0) = 0$ (this will not extend to an analytic function, of course, but that does not matter). Now we must determine how the boundary condition changes; in other words, if we define $h : \partial D \to \mathbf{R}$ by

$$h = \begin{cases} \sin 2\theta, & r = 1 \\ r^2, & \theta = \pi/4 \\ -r^2, & \theta = -\pi/4 \end{cases},$$

then we need to determine $h \circ f : \partial E \to \mathbf{R}$. Now ∂E also has three pieces, namely $r = 1$, $\theta \in [-\pi/2, \pi/2]$, $\theta = \pi/2$, $r \in [0, 1]$, and $\theta = -\pi/2$, $r \in [0, 1]$. Now if $z = re^{i\theta}$, where $\theta \in (-\pi, \pi)$, then $f(z) = r^{1/2} e^{\frac{1}{2} i\theta}$; thus these three boundary pieces are mapped to, respectively,

$$r = 1,\, \theta \in [-\pi/4, \pi/4]; \quad \theta = \pi/4,\, r \in [0, 1]; \quad \theta = -\pi/4,\, r \in [0, 1].$$

Now if $r = 1$ and $\theta \in [-\pi/2, \pi/2]$, then we have $r^{1/2} = 1$ and

$$(h \circ f)(re^{i\theta}) = h(r^{1/2} e^{\frac{1}{2} i\theta}) = h(e^{\frac{1}{2} i\theta}) = \sin 2 \cdot \left(\frac{1}{2}\theta\right) = \sin\theta;$$

if $\theta = \pi/2$ and $r \in [0, 1]$, then we have $\frac{1}{2}\theta = \pi/4$ and

$$(h \circ f)(re^{i\theta}) = h(r^{1/2} e^{\frac{1}{2} i\theta}) = \left(r^{1/2}\right)^2 = r;$$

finally, if $\theta = -\pi/2$ and $r \in [0, 1]$, then we have $\frac{1}{2}\theta = -\pi/4$ and

$$(h \circ f)(re^{i\theta}) = h(r^{1/2} e^{\frac{1}{2} i\theta}) = -\left(r^{1/2}\right)^2 = -r.$$

Pulling this together, we have

$$h \circ f = \begin{cases} \sin\theta, & r = 1 \\ r, & \theta = \pi/2 \\ -r, & \theta = -\pi/2 \end{cases},$$

so that we wish to solve the problem

$$\Delta v = 0 \text{ on } E, \qquad v|_{\partial E} = \begin{cases} \sin\theta, & r = 1 \\ r, & \theta = \pi/2 \\ -r, & \theta = -\pi/2 \end{cases} \tag{5}$$

Now we note that if a, b, and c are real numbers, then the function

$$g(x, y) = a + bx + cy \tag{6}$$

will be harmonic at every point $(x, y) \in \mathbf{R}^2$, since all of its second order derivatives vanish. Let us see whether we can find a solution to (5) which is of this form. The idea is that, since all of these functions satisfy Laplace's equation, we only need to fit the boundary conditions. Now in polar coordinates, the function g may be written

$$g = a + br\cos\theta + cr\sin\theta.$$

If $r = 1$, this gives $a + b\cos\theta + c\sin\theta$, which will fit the boundary condition in (5) if $a = b = 0$, $c = 1$; if $\theta = \pm\pi/2$, then it gives

$$a \pm cr,$$

which will also fit the boundary condition in (5) if $a = 0$ and $c = 1$. In other words, we have found that the function

$$v(x, y) = y$$

is a solution to (5).

Working backwards, then, we know from our general work above that the function $u = v \circ f^{-1}$ will solve our original problem. Now $f^{-1}(z) = z^2$, so that we have finally the solution

$$u(x, y) = v[f(x + iy)] = v(x^2 - y^2, 2xy) = 2xy$$

to the boundary value problem (4).

If we step back and look at the broad sweep of the logic used in this example, we see that to make efficacious use of this technique in practice, we must have a catalogue of two things: one, transformations and regions; two, standard solutions to Laplace's equation on the domain regions of these transformations. For us, the main class of standard solutions will be the linear ones given in (6). We now describe a few elementary transformations which are useful.

EXAMPLES of transformations and regions useful for solving Laplace's equation.

1. Powers and roots on wedges. If $n \in \mathbf{Z}$, $n > 0$, then the map $z \mapsto z^n$ takes a wedge with vertex at the origin with angle α to another wedge with vertex at the origin with angle $n\alpha$. For this transformation to be invertible, the angle α must be less than $2\pi/n$. Similarly, if $n \in \mathbf{Z}$, $n > 0$, then the map $z \mapsto z^{1/n}$, after taking an appropriate branch, will map a wedge of angle α with vertex at the origin to another wedge with vertex at the origin and angle α/n, assuming that $\alpha < 2\pi$. Wedges have three boundaries, two of which are lines and the third of which is a circular arc; on the circular arc, a map $z \mapsto z^p$ ($p = n$ or $p = 1/n$, as the case may be) will take a point $re^{i\theta}$ to a point $r^p e^{ip\theta}$, i.e., it will multiply the angle θ by p (where θ must be in the range corresponding to the chosen branch if $p = 1/n$), while on the lines the map acts by simple exponentiation, as in the example above.

2. Inversion. The map $z \mapsto z^{-1}$, as we have seen before (see §16 of these lecture notes), takes the punctured plane into itself, but 'turns it inside out' by mapping the region outside the unit circle to the region inside the unit circle.

3. The exponential function. Consider the map $z \mapsto e^z$. We have seen already (see the solutions to the first homework assignment) that this map takes lines parallel to the real axis to rays from the origin, and lines parallel to the imaginary axis to circles centred at the origin. Thus this map will take certain rectangular regions in the plane into annular wedges. (Going further, to whole annular regions or even a full disk, would involve extra considerations beyond the scope of our present discussion.)

4. The maps $z \mapsto \sin z$ and $z \mapsto \cos z$. The second of these has already been described in some detail in §16 of these lecture notes (cf. Goursat, §22); in particular, it was shown there that $z \mapsto \cos z$ takes the rectangle $\{x + iy \,|\, x \in (0, \pi),\ y \in (0, +\infty)\}$ into the lower half-plane $\{x + iy \,|\, y < 0\}$. Let us see a bit more carefully what it does on the boundary of this rectangle. This boundary consists of three lines, the half-line $x = 0$, $y \geq 0$, the line segment $y = 0$, $x \in [0, \pi]$, and the half-line $x = \pi$, $y \geq 0$. Let us consider these in turn. We have (see, e.g., §13 of these notes)

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y;$$

thus if $x = 0$ we have

$$\cos(iy) = \cosh y,$$

which takes the interval $[0, +\infty)$ into the interval $[1, +\infty)$ in a one-to-one fashion. Similarly, if $x = \pi$, then we have

$$\cos(\pi + iy) = -\cosh y,$$

which takes the interval $[0, +\infty)$ into the interval $(-\infty, -1]$. Finally, if $y = 0$, $\cos z$ will be just the ordinary real-valued function $\cos x$, which takes the interval $[0, \pi]$ into the interval $[-1, 1]$ (though in reverse order, i.e., it is decreasing on that interval). All told, then, the half-line $x = 0$, $y \geq 0$ will map to the interval $[1, +\infty)$, the segment $y = 0$, $x \in [0, \pi]$ will map to the interval $[-1, 1]$ (where the point 1 is mapped from the intersection $x = y = 0$ of these two parts of the boundary) and finally the half-line $x = \pi$, $y \geq 0$ will map to the interval $(-\infty, -1]$, with as before the point -1 is mapped from the intersection $x = \pi$, $y = 0$ of these last two parts of the boundary. This suggests that this map may be useful if we are interested in finding solutions to problems on the lower half-plane whose initial data can be broken down in some way across the three intervals $(-\infty, -1]$, $[-1, 1]$, and $[1, +\infty)$.

We note one other result: if $x = \pi/2$, then we have

$$\cos(x + iy) = \cos(\pi/2 + iy) = -i \sinh y,$$

from which we see that the half-line $x = \pi/2$, $y \geq 0$ is mapped to the negative imaginary axis. This means, incidentally, that the two rectangles

$$\{x + iy \,|\, x \in (0, \pi/2),\ y > 0\}, \quad \{x + iy \,|\, x \in (\pi/2, \pi),\ y > 0\}$$

map to the fourth and third quadrants, respectively (since $\cos x > 0$ for $x \in (0, \pi/2)$ and $\cos x < 0$ for $x \in (0, \pi/2)$). Thus this map can also be used for problems on a quarter-plane.

Similarly, let us consider the map $z \mapsto \sin z$. We have (ibid.)

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y.$$

In this case we take as our domain the rectangular strip $\{x + iy \,|\, x \in (-\pi/2, \pi/2),\ y > 0\}$, since $\sin x$ is invertible on $(-\pi/2, \pi/2)$ but not on any strictly larger interval. Let us consider the values of sin on the three boundary lines as we did for cos. If $x = -\pi/2$, then we have

$$\sin(-\pi/2 + iy) = -\cosh y,$$

while if $x = \pi/2$ then we have

$$\sin(\pi/2 + iy) = \cosh y;$$

thus these two lines map to the segments $(-\infty, -1]$ and $[1, +\infty)$, similarly to what we found for cos (except note that the order is reversed). Similarly, if $y = 0$ then $\sin z$ is just the ordinary real-valued sine function $\sin x$, which maps the interval $[-\pi/2, \pi/2]$ to the interval $[-1, 1]$. Thus, again, the boundary is mapped onto the entire real line. Since $\cos x$ and $\sinh y$ are both positive everywhere on the rectangle, we see that sin maps the rectangle into the upper half-plane; and this map is actually onto. Again, as with cos, we see that the midline $x = 0$, $y \geq 0$ maps to the positive real axis, and we see further that the rectangles

$$\{x + iy \,|\, x \in (-\pi/2, 0),\ y > 0\}, \quad \{x + iy \,|\, x \in (0, \pi/2)\ y > 0\}$$

map to the second and first quadrants, respectively (again, which quadrant is the image of which rectangle can be determined from the sign of x).