Summary:

- We motivate the definition of the Poisson kernel.

In lecture we attempted to give a *derivation* of the Poisson kernel. While the main idea was correct, there were a few errors in detail and interpretation, and when those are corrected it turns out that what was given can *motivate* the definition of the Poisson kernel, but does not really serve as a proof. We go through this motivation anyway.

Let D be the disk of radius r centred at the origin, $C = \partial D$ its boundary (the unit circle centred at the origin), and consider the following problem:

$$\Delta u = 0, \qquad u|_{\partial D} = h, \tag{1}$$

where Δ is the Laplacian, $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$, and h is a function on C. Now suppose that there is a function f which is analytic on the complex plane and satisfies $\operatorname{Re} f|_{\partial D} = h$; since $\operatorname{Re} f$ must be harmonic everywhere on the plane, and in particular on D, we see that $u = \operatorname{Re} f$ is a solution to problem (1). Now the Cauchy integral formula allows us to write

$$f(x+iy) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z' - [x+iy]}\,dz'. \tag{2}$$

If we define

$$z^* = \frac{r^2}{\overline{z}}$$

[note that this corrects an error in the lecture, where z^* was mistakenly given as r^2/z], then z^* will be outside of C so that we will have

$$\int_C \frac{f(x'+iy')}{z'-z^*}\,dz' = 0 \tag{3}$$

Thus we may subtract this integral from (2). Now we may parameterise C as

$$z'(t) = re^{i\theta}, \qquad \theta \in [0, 2\pi];$$

let us write also $x + iy = r_0 e^{i\theta_0}$ for some θ_0. Then (2) becomes, after subtracting (3),

$$\begin{aligned}
f(x+iy) &= \frac{1}{2\pi i}\int_0^2 \pi \left[\frac{1}{re^{i\theta} - r_0 e^{i\theta_0}} - \frac{1}{re^{i\theta} - \frac{r^2}{r_0}e^{i\theta_0}}\right] f(re^{i\theta}) ire^{i\theta}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{-\frac{r^2}{r_0}e^{i\theta_0} + r_0 e^{i\theta_0}}{(re^{i\theta} - r_0e^{i\theta_0})\left(re^{i\theta} - \frac{r^2}{r_0}e^{i\theta_0}\right)} re^{i\theta} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{-(r^2 - r_0^2)\frac{r}{r_0}e^{i(\theta_0+\theta)}}{(re^{i\theta} - r_0e^{i\theta_0})\left(re^{i\theta} - \frac{r^2}{r_0}e^{i\theta_0}\right)} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{-\frac{r^2 - r_0^2}{r_0^2}e^{i(\theta_0+\theta)}}{\left(\frac{r}{r_0}e^{i\theta} - e^{i\theta_0}\right)\left(e^{i\theta} - \frac{r}{r_0}e^{i\theta_0}\right)} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\frac{r^2 - r_0^2}{r_0^2}}{\left(\frac{r}{r_0} - e^{i(\theta_0-\theta)}\right)\left(\frac{r}{r_0} - e^{i(\theta-\theta_0)}\right)} f(re^{i\theta})\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{\left|r - r_0e^{i(\theta_0-\theta)}\right|^2} f(re^{i\theta})\,d\theta.
\end{aligned}$$

If we take the real part of this, then, since everything in the integrand is real except for f, and $\operatorname{Re} f|_{\partial D} = h$, we have

$$\begin{aligned}
u(x,y) = \operatorname{Re} f &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{\left|r - r_0e^{i(\theta_0-\theta)}\right|^2} h(r\cos\theta, r\sin\theta)\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)} h(r\cos\theta, r\sin\theta)\,d\theta.
\end{aligned} \tag{4}$$

As noted above, the foregoing does not actually *prove* that given a continuous function h the above function u will give a solution to problem (1); however, this can be proved by other means, though we shall not do so here. We shall however give an example.

Example. Let us start with a trivial example:

$$\Delta u = 0, \qquad u|_{\partial D} = 1.$$

Clearly the solution to this is 1. Using the integral formula (4), we have

$$\begin{aligned} u(x,y) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos\theta}\,d\theta. \end{aligned} \tag{5}$$

Now it turns out that the integrand here has an explicit antiderivative. To determine it, we work with the integral

$$\int \frac{1}{a - b\cos\theta}\,d\theta, \tag{6}$$

where we assume $a > b \geq 0$. Note that, since $\cos\theta = \cos^2\theta/2 - \sin^2\theta/2 = 2\cos^2\theta/2 - 1$,

$$a - b\cos\theta = (a+b) - b(1 + \cos\theta) = (a+b) - 2b\cos^2\frac{\theta}{2},$$

so that the integral (6) may be rewritten as

$$\int \frac{1}{(a+b) - 2b\cos^2\frac{\theta}{2}}\,d\theta = \int \frac{\sec^2\frac{\theta}{2}}{(a+b)\sec^2\frac{\theta}{2} - 2b}\,d\theta.$$

Let us now make the substitution $v = \tan\theta/2$, $dv = \frac{1}{2}\sec^2\theta/2\,d\theta$; then this integral becomes, since $\sec^2 x = 1 + \tan^2 x$,

$$\begin{aligned} 2\int \frac{1}{(a+b)(1+v^2) - 2b}\,dv &= \frac{2}{a+b}\int \frac{1}{v^2 + \frac{a-b}{a+b}}\,dv \\ &= \frac{2}{a+b}\cdot\left[\frac{a+b}{a-b}\right]^{1/2}\tan^{-1}\left[\left\{\frac{a+b}{a-b}\right\}^{1/2} v\right] \\ &= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\left\{\frac{a+b}{a-b}\right\}^{1/2} v\right], \end{aligned}$$

from which we obtain finally

$$\int \frac{1}{a - b\cos\theta}\,d\theta = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\left\{\frac{a+b}{a-b}\right\}^{1/2}\tan\frac{\theta}{2}\right].$$

Now for us $a = r^2 + r_0^2$ while $b = 2rr_0$, so $a + b = (r + r_0)^2$, $a - b = (r - r_0)^2$, and $\sqrt{a^2 - b^2} = r^2 - r_0^2$, and we obtain

$$\int \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos\theta}\,d\theta = 2\tan^{-1}\left[\frac{r + r_0}{r - r_0}\tan\frac{\theta}{2}\right].$$

Note that this formula is only valid on intervals which do not contain any odd integer multiple of π; for example, $(-\pi,\pi)$, $(\pi, 3\pi)$, and so on. This is because $\tan\frac{\theta}{2}$ is not defined at odd integer multiples of π. Thus to evaluate our original integral (5) we must split the interval $[0, 2\pi]$ into two pieces, $[0,\pi)$ and $(\pi, 2\pi]$, and treat each piece as an improper integral. The two integrals we thus get are

$$\begin{aligned} \frac{1}{2\pi}\int_0^{\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)}\,d\theta &= \frac{1}{2\pi}\lim_{\theta\to\pi^-} 2\tan^{-1}\left[\frac{r + r_0}{r - r_0}\tan\frac{\theta}{2}\right]\Bigg|_0^{\theta} = \frac{1}{2\pi}(\pi - 0) = \frac{1}{2}, \\ \frac{1}{2\pi}\int_{\pi}^{2\pi} \frac{r^2 - r_0^2}{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)}\,d\theta &= \frac{1}{2\pi}\lim_{\theta\to\pi^+} 2\tan^{-1}\left[\frac{r + r_0}{r - r_0}\tan\frac{\theta}{2}\right]\Bigg|_{\theta}^{2\pi} = \frac{1}{2\pi}(0 - (-\pi)) = \frac{1}{2}, \end{aligned}$$

and finally we obtain $u = 1$, as we found originally by inspection.