Summary:
- We demonstrate additional methods of computing definite integrals using contours.
- We then proceed to discuss more general properties of analytic functions.
- In particular, we prove Liouville's Theorem, the argument principle, and Rouché's Theorem, and use these to give two different proofs of the fundamental theorem of algebra.
- We then discuss the Poisson kernel for Laplace's equation on a disk.

(Goursat, §§36, 45, 48 – 49.)

35. Additional methods of closing the contour. We show two more methods of closing the contour on a definite integral, by way of example.

EXAMPLE. Evaluate the integral

$$\int_0^\infty \frac{x^{1/3}}{(1+x^2)^2}\,dx. \tag{1}$$

We note first of all that this integral converges (apply the usual power test). Now to extend this to a contour integral over a complex curve we must first choose a branch of the cube root function. We shall choose the branch with a branch cut along the positive real axis*, and take the angle θ to lie in the interval $(0, 2\pi)$. To evaluate integral (1), we consider a so-called *keyhole contour* composed of four separate curves (see the figure): a line L_R from $i\epsilon$ to $R + i\epsilon$; a circular arc C_R running counterclockwise from $R + i\epsilon$ to $R - i\epsilon$, centred at the origin; another line L'_R running from $R - i\epsilon$ to $-i\epsilon$; and finally a semicircle C'_ϵ running clockwise from $-i\epsilon$ to $i\epsilon$ in the third and second quadrants, again centred at the origin. (There are of course other slightly different contours which would perform the same task equally well.) Let us let $\gamma = L_R + C_R + L'_R + C'_\epsilon$ denote the full curve. Note that since the branch point and branch cut of $z^{1/3}$ lie entirely outside of this curve (this is our first indication that taking a branch cut along the line of integration was in fact the correct thing to do!) the integrand $z^{1/3}/(1 + z^2)^2$ has only poles within the contour, at $z = \pm i$, we see from the residue theorem that

$$\int_\gamma \frac{z^{1/3}}{(1+z^2)^2}\,dz = 2\pi i\left[\mathrm{Res}_i \frac{z^{1/3}}{(1+z^2)^2} + \mathrm{Res}_{-i}\frac{z^{1/3}}{(1+z^2)^2}\right].$$

We will come back to the calculation of these residues later and consider first how the integral at left relates to the integral in (1). In particular, we claim that

$$\int_{C_R} \frac{z^{1/3}}{(1+z^2)^2}\,dz \to 0 \quad\text{and}\quad \int_{C'_\epsilon} \frac{z^{1/3}}{(1+z^2)^2}\,dz \to 0 \quad\text{as}\quad R\to\infty,\ \epsilon\to 0^+. \tag{2}$$

Note first that if z is on C_R or C'_ϵ, then $|z^{1/3}| = |z|^{1/3}$, where $|z|^{1/3}$ denotes the unique positive cube root of the positive real number z (positive since the curves C_R and C'_ϵ do not pass through the origin). Thus, parameterising C_R as Re^{it}, $t \in [\theta_0, 2\pi - \theta_0]$, we have (note that the equation $|z^{1/3}| = |z|^{1/3}$ does not depend on which branch of the cube root function is used to calculate $z^{1/3}$, and thus we do not need to worry here and in the next integral whether the point as parameterised has an angle lying within the interval $(0, 2\pi)$)

$$\begin{aligned}\left|\int_{C_R} \frac{z^{1/3}}{(1+z^2)^2}\,dz\right| &\le \int_{\theta_0}^{2\pi-\theta_0} \frac{R^{1/3}}{|1+R^2e^{2it}|^2}R\,dt \\ &\le \int_{\theta_0}^{2\pi-\theta_0} \frac{R^{4/3}}{(R^2-1)^2}\,dt = (2\pi - 2\theta_0)\frac{R^{4/3}}{(R^2-1)^2} \to 0\end{aligned}$$

as $R\to\infty$, since the degree of the denominator is greater than that of the numerator. Similarly, parameterising C'_ϵ as ϵe^{-it}, $t\in[\pi/2, 3\pi/2]$, we have

$$\begin{aligned}\left|\int_{C'_\epsilon} \frac{z^{1/3}}{(1+z^2)^2}\,dz\right| &\le \int_{\pi/2}^{3\pi/2} \frac{\epsilon^{1/3}}{|1+\epsilon^2 e^{-2it}|^2}\epsilon\,dt \\ &\le \int_{\pi/2}^{3\pi/2} \frac{\epsilon^{4/3}}{(1+\epsilon^2)^2}\,dt = \pi\frac{\epsilon^{4/3}}{(1+\epsilon^2)^2} \to 0\end{aligned}$$

* It may seem like a very bad idea to take the branch cut along the positive real axis as, after all, this is exactly the line along which we wish to integrate! As we shall see shortly, though, this is exactly the right place to put the branch cut in this case.

as $\epsilon \to 0^+$, since the denominator goes to 1 and the numerator to 0. This completes the demonstration of (2).

We thus only need to consider how the integrals over L_R and L'_R relate to (1). We parameterise L_R by $t + i\epsilon$ and L'_R by $(R-t) - i\epsilon$, where in both cases $t \in [0, R]$. Since $\epsilon > 0$, we may express these numbers in polar notation as (here arctan gives the angle in $(-\pi/2, \pi/2)$ which has the given tangent)

$$t + i\epsilon = \sqrt{t^2 + \epsilon^2} e^{i \arctan \epsilon/t}, \tag{3}$$

$$\begin{aligned}(R-t) - i\epsilon &= \sqrt{(R-t)^2 + \epsilon^2} e^{i \arctan(-\epsilon/(R-t))} = \sqrt{(R-t)^2 + \epsilon^2} e^{-i \arctan \epsilon/(R-t)} \\ &= \sqrt{(R-t)^2 + \epsilon^2} e^{i(2\pi - \arctan \epsilon/(R-t))},\end{aligned} \tag{4}$$

where the 2π in (4) was added to make the angle lie in the interval $(0, 2\pi)$ corresponding to our chosen branch of $z^{1/3}$. Given this, then, the integrals over L_R and L'_R become

$$\int_{L_R} \frac{z^{1/3}}{(1+z^2)^2} \, dz = \int_0^R \frac{(t^2 + \epsilon^2)^{1/6} e^{i \frac{1}{3} \arctan \frac{\epsilon}{t}}}{(1 + (t + i\epsilon)^2)^2} \, dt,$$

$$\int_{L'_R} \frac{z^{1/3}}{(1+z^2)^2} \, dz = -\int_0^R \frac{[(R-t)^2 + \epsilon^2]^{1/6} e^{i \frac{1}{3}(2\pi - \arctan \frac{\epsilon}{R-t})}}{(1 + (R - t - i\epsilon)^2)^2} \, dt.$$

If we take the limit of these expressions as $\epsilon \to 0^+$ and interchange it with the integrals, we obtain, since $\arctan 0 = 0$,

$$\lim_{\epsilon \to 0^+} \int_{L_R} \frac{z^{1/3}}{(1+z^2)^2} \, dz = \int_0^R \frac{t^{1/3}}{(1+t^2)^2} \, dt,$$

$$\lim_{\epsilon \to 0^+} \int_{L'_R} \frac{z^{1/3}}{(1+z^2)^2} \, dz = -\int_0^R \frac{(R-t)^{1/3} e^{\frac{2\pi i}{3}}}{(1 + (R-t)^2)^2} \, dt = -e^{\frac{2\pi i}{3}} \int_0^R \frac{t^{1/3}}{(1+t^2)^2} \, dt,$$

and we see finally that

$$\lim_{R \to \infty} \lim_{\epsilon \to 0^+} \int_\gamma \frac{z^{1/3}}{(1+z^2)^2} \, dz = \left(1 - e^{\frac{2\pi i}{3}}\right) \int_0^\infty \frac{x^{1/3}}{(1+x^2)^2} \, dx,$$

so that the integral (1) is equal to

$$\int_0^\infty \frac{x^{1/3}}{(1+x^2)^2} \, dx = \frac{2\pi i}{1 - e^{\frac{2\pi i}{3}}} \left[\operatorname{Res}_i \frac{z^{1/3}}{(1+z^2)^2} + \operatorname{Res}_{-i} \frac{z^{1/3}}{(1+z^2)^2} \right]. \tag{5}$$

We are thus left with the task of computing the residues. Note that both $\pm i$ are poles of order 2 of the function $z^{1/3}/(1+z^2)^2$; thus letting $\lambda = \pm i$, we may write

$$\operatorname{Res}_\lambda \frac{z^{1/3}}{(1+z^2)^2} = \lim_{z \to \lambda} \frac{d}{dz} (z - \lambda)^2 \frac{z^{1/3}}{(1+z^2)^2}.$$

Since $1 + z^2 = (z - i)(z + i)$, this allows us to write

$$\begin{aligned}\operatorname{Res}_i \frac{z^{1/3}}{(1+z^2)^2} &= \lim_{z \to i} \frac{d}{dz} \frac{z^{1/3}}{(z+i)^2} = \lim_{z \to i} \frac{\frac{1}{3z^{2/3}}(z+i)^2 - 2(z+i)z^{1/3}}{(z+i)^4} \\ &= \frac{\frac{-4}{3e^{i\pi/3}} - 4ie^{i\pi/6}}{16} = -\frac{1}{4}\left[\frac{1}{3}e^{-i\pi/3} + ie^{i\pi/6}\right] \\ &= -\frac{1}{4}\left[\frac{1}{3}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + i\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\right] = -\frac{1}{4}\left[\left(\frac{1}{3} - 1\right)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right] \\ &= \frac{1}{6}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right), \\ \operatorname{Res}_{-i} \frac{z^{1/3}}{(1+z^2)^2} &= \lim_{z \to -i} \frac{d}{dz} \frac{z^{1/3}}{(z-i)^2} = \frac{1}{16}\left[\frac{1}{3e^{i\pi}}(-4) + 4ie^{i\pi/2}\right] \\ &= \frac{1}{4}\left(\frac{1}{3} - 1\right) = -\frac{1}{6},\end{aligned}$$

so

$$\mathrm{Res}_i \frac{z^{1/3}}{(1+z^2)^2} + \mathrm{Res}_{-i} \frac{z^{1/3}}{(1+z^2)^2} = \frac{1}{6}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{1}{6}e^{i\pi/3}.$$

At this point we may start to feel like something has gone wrong since our answer at the end of the day needs to be a real number, but it is not clear how we shall get a real number out of what we have so far. Everything does work out in the end, though: by (5) we have

$$\begin{aligned}\int_0^\infty \frac{x^{1/3}}{(1+x^2)^2}\,dx &= \frac{2\pi i}{1-e^{\frac{2\pi i}{3}}}\left[-\frac{1}{6}e^{\frac{i\pi}{3}}\right] \\ &= -\frac{\pi}{6}\frac{2i}{e^{-i\pi/3} - e^{i\pi/3}} = \frac{\pi}{6}\sin\frac{\pi}{3} = \frac{\pi\sqrt{3}}{12},\end{aligned}$$

which is thus our final answer.

[Deep breath!]

Note how taking the branch cut along the line of integration helped us: since branch points are singularities which are not poles, whatever closed contour we draw in the complex plane must exclude the branch point and the branch cut; and as we bring the edges of the keyhole contour (the lines L_R and L'_R) together, the discontinuity of $z^{1/3}$ across the branch cut will allow us to combine the two integrals without cancellation, to get a multiple of the integral along the branch cut. Thus it is precisely a multiple of the integral along the branch cut which is equal to the sum of the residues of the integrand at its poles within the contour.

Let us consider one more way of closing the contour.

Example. Evaluate

$$\int_0^\infty \frac{1}{1+x^3}\,dx. \tag{6}$$

This integral is much simpler than the previous one! It also demonstrates a slightly different application of the type of 'wedge contour' we saw when evaluating the Fresnel integrals previously. We start out with the line L_R which is just the interval $[0, R]$ in this case (since we have no branch cut!). We then consider closing the contour using a circular arc C_R of angle α followed by a line L'_R back to the origin (necessarily therefore making an angle α with the positive real axis). Now we wish the integral over L'_R to be related somehow to the original integral (6). Parameterising it as $(R-t)e^{i\alpha}$, we have

$$\int_{L'_R} \frac{1}{1+z^3}\,dz = -\int_0^R \frac{1}{1+(R-t)^3e^{3i\alpha}}e^{i\alpha}\,dt = -\int_0^R \frac{1}{1+t^3e^{3i\alpha}}e^{i\alpha}\,dt.$$

For this to be a multiple of (6), we must have $3\alpha = 2n\pi$ for some integer n. Since we wish also to include as few residues as possible in the closed contour, we want α to be as small as possible, and therefore take $\alpha = 2\pi/3$. Thus we will close using the contour shown in the figure. Now since $R/|1+z^3| \to 0$ as $R \to \infty$, when $z \in C_R$, we must have

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{1+z^3}\,dz = 0.$$

Now the only poles of the integrand in (6) are at the complex cube roots of -1, which are $e^{i\pi/3}$, $e^{i\pi}$, and $e^{5\pi/3}$; only the first of these lies within the contour $L_R + C_R + L'_R$, and thus we may write

$$(1-e^{2\pi i/3})\int_0^\infty \frac{1}{1+x^3}\,dx = 2\pi i \mathrm{Res}_{\pi i/3}\frac{1}{1+z^3}. \tag{7}$$

Now this pole is simple, so the residue can be calculated as follows:

$$\mathrm{Res}_{\pi i/3}\frac{1}{1+z^3} = \lim_{z\to\pi i/3}\frac{z-\pi i/3}{1+z^3} = \lim_{z\to\pi i/3}\frac{z-\pi i/3}{(1+z^3)-(1+[e^{i\pi/3}]^3)} = \left(\frac{d}{dz}(1+z^3))\Big|_{z=e^{\pi i/3}}\right)^{-1};$$

compare our result on p. 6 of the lecture notes for July 14 – 16. This evaluates to

$$\frac{1}{3}e^{-2\pi i/3},$$

and so by (7) we have

$$\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi i e^{-2\pi i/3}}{3(1-e^{2\pi i/3})} = \frac{\pi}{3}\frac{2i}{e^{-\pi i/3}-e^{\pi i/3}}e^{-\pi i} = \frac{\pi}{3}\sin\frac{\pi}{3} = \frac{\pi\sqrt{3}}{6}.$$

That this integral is exactly twice that in the previous example, is a complete coincidence. (As far as I know! – I picked both integrals out of a hat, pretty much.)

The idea in this method can be combined with that in the previous example – i.e., we can make a contour consisting of two lines at angles to each other, such that the integrals over these lines can be written in terms of each other, as well as a large circle and a small circle around the origin if that happens to be a branch point. This idea is useful on question 2 of the August 3 – 7 homework.

There is one more method for turning definite integrals into contour integrals which we shall have use for; see Goursat, §45. We, again, show this by way of an example.

EXAMPLE. Evaluate the integral

$$\int_0^{2\pi} \frac{1}{2-\sin x}\,dx. \tag{8}$$

We begin by noting that the integrand is continuous on the real line, so that the integral is defined. Note that this integral is fundamentally different from the other integrals we have studied so far, since it is taken over a finite interval instead of an infinite one. Thus it does not seem that the method of considering it as a contour integral and then closing the contour, as we have done previously, will be of use here. We shall instead do something completely different: rewrite (8) as a contour integral by, effectively, *deparameterising* it: in other words, recognising integral (8) as the parameterised form of an integral over a closed contour in the complex plane.

To do this, note first of all that

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i};$$

now as x ranges from 0 to 2π, e^{ix} and e^{-ix} both trace out the unit circle. Thus it would seem that integral (8) may be the parameterisation of an integral over the unit circle. Now the only other part of the integrand which depends on x is dx; if we let $z = e^{ix}$, then $dz = ie^{ix}dx = iz dx$, so $dx = dz/(iz)$ (note that since z is on the unit circle, $z \neq 0$ so $1/z$ is defined), and we have finally that (letting C denote the unit circle)

$$\int_0^{2\pi} \frac{1}{2-\sin x}\,dx = \int_C \frac{1}{2-\frac{z-z^{-1}}{2i}}\frac{dz}{iz},$$

since if $z = e^{ix}$ then $e^{-ix} = 1/z$. Now we may simplify this integral as follows:

$$\int_C \frac{1}{2-\frac{z-z^{-1}}{2i}}\frac{dz}{iz} = \int_C \frac{2}{4iz-z^2+1}\,dz = -2\int_C \frac{1}{z^2-4iz-1}\,dz. \tag{9}$$

This is exactly the kind of integral which can be evaluated using the residue theorem! We just need to find the poles of the integrand. Now $z^2-4iz-1=0$ can be solved using the quadratic formula:

$$z = 2i + (-4+1)^{1/2} = 2i + i\sqrt{3}, 2i - i\sqrt{3} = i(2\pm\sqrt{3}).$$

Now $\sqrt{3} \in (1,2)$, so that the root $i(2+\sqrt{3})$ lies outside the unit circle while the other root, $i(2-\sqrt{3})$, lies within it. Thus the integral (9) can be evaluated by computing the residue of the integrand at this root, which is

$$\begin{aligned}\mathrm{Res}_{i(2-\sqrt{3})}\frac{1}{z^2-4iz-1} &= \lim_{z\to i(2-\sqrt{3})}\frac{z-i(2-\sqrt{3})}{z^2-4iz-1} = \lim_{z\to i(2-\sqrt{3})}\frac{1}{z-i(2+\sqrt{3})} \\ &= \frac{1}{-2i\sqrt{3}} = \frac{i}{2\sqrt{3}},\end{aligned}$$

so that finally

$$\int_0^{2\pi} \frac{1}{2-\sin x}\,dx = -4\pi i \cdot \frac{i}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{3}.$$

Again, that this integral is *also!* an integral multiple of the previous too, is still a complete coincidence!

The method in this last example can be adapted to many other integrals with integrands which are rational functions of $\sin x$ and $\cos x$, defined everywhere on the interval of integration. See Goursat, §45, for a general discussion.

36. Liouville's Theorem. We prove the following result:

Liouville's Theorem. Let f be a function which is analytic and bounded on the entire complex plane. Then f must be constant.

This means that there must be a constant M such that $|f(z)| \le M$ for all $z \in \mathbf{C}$. The proof is an easy application of the Cauchy integral formula. Let $z \in \mathbf{C}$, let $R > 0$ be any positive real number, and let C_R denote the circle of radius R centred at z. Then we must have

$$f'(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z')}{(z'-z)^2}\,dz',$$

so

$$|f'(z)| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(z')|}{R^2} R\,dt \le \frac{1}{2\pi}\int_0^{2\pi} \frac{M}{R}\,dt = \frac{M}{R}.$$

But now $f'(z)$ cannot depend on R; since the right-hand side of the above expression goes to zero as $R \to \infty$, we must have $|f'(z)| = 0$, i.e., $f'(z) = 0$, for all $z \in \mathbf{C}$. This means that f must itself be constant, as claimed.

[I can't remember if we have ever proved that f' identically zero means that f must be constant when f is an analytic function. At any rate it is not hard to prove. Suppose that f is analytic on the interior of some simple closed curve, and pick any point z_0 inside that curve; then we can write, by the fundamental theorem of calculus,

$$f(z) = \int_{z_0}^{z} f'(z')\,dz' + f(z_0) = f(z_0),$$

since $f'(z') = 0$. This means that f must be constant.]

This can be used to prove that every (nonconstant) polynomial with complex coefficients has at least one complex root. To see this, let

$$P(z) = a_n z^n + \cdots + a_0,$$

where a_n, $\cdots$, $a_0 \in \mathbf{C}$ and we assume that $a_n \ne 0$. Suppose that P has no complex roots; we shall show that this implies that $n = 0$, so that P is constant. Since P has no complex roots, its reciprocal $1/P$ must be analytic everywhere on the complex plane. Now note that

$$\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n} \tag{10}$$

must go to zero as $|z|$ goes to infinity; thus there is an $R > 0$ such that the modulus of (10) is less than $\frac{1}{2}|a_n|$ when $|z| \ge R$. For such z, then, we have

$$\begin{aligned}|P(z)| &= |z|^n \left|a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right| \ge R^n \left[|a_n| - \left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right|\right] \\ &\ge R^n \left[|a_n| - \frac{1}{2}|a_n|\right] = \frac{1}{2}R^n |a_n|,\end{aligned}$$

so since $a_n \ne 0$ we must have

$$\left|\frac{1}{P(z)}\right| \le \frac{2}{R^n |a_n|}$$

for $|z| \ge R$. Thus $1/P$ is bounded on the exterior of the disk of radius R centred at the origin. But since $|1/P|$ is a continuous function, and this disk is closed and bounded, $|1/P|$ must be bounded inside the disk as well; thus it must be bounded everywhere, and by Liouville's Theorem it must therefore be constant. Since it cannot be equal to zero, it must equal a nonzero constant, and hence $P = 1/(1/P)$ must be constant as well, as claimed.

37. The argument principle and Rouché's Theorem. We prove the following result:

The Argument Principle. Let C be a simple closed curve, and let f be a function analytic within and on C except possibly for poles within C, and which is nonzero on C. If Z denotes the number of zeroes and P the number of poles of f within C, counted with multiplicity, then

$$\int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i(Z-P).$$

To see this, we note the following result. If z_0 is a zero of f within C, say of multiplicity m, then there is a nonzero analytic function ϕ on a disk around z_0 such that near z_0

$$f(z) = (z-z_0)^m\phi(z);$$

while if z_0 is a pole of f within C, say of order n, then there is a nonzero analytic function ψ on a disk around z_0 such that near z_0

$$f(z) = (z-z_0)^{-n}\psi(z).$$

In the first case,

$$\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1}\phi(z) + (z-z_0)^m\phi'(z)}{(z-z_0)^m\phi(z)} = \frac{m}{z-z_0} + \frac{\phi'(z)}{\phi(z)},$$

and we note that ϕ'/ϕ is analytic since ϕ is nonzero. Similarly, in the second case

$$\frac{f'(z)}{f(z)} = \frac{-n(z-z_0)^{-n-1}\psi(z) + (z-z_0)^{-n}\psi'(z)}{(z-z_0)^{-n}\psi(z)} = -\frac{n}{z-z_0} + \frac{\psi'(z)}{\psi(z)},$$

and again ψ'/ψ is analytic since ψ is nonzero. Now if z_0 is a point in C which is neither a pole nor a zero of f, then clearly f'/f is itself analytic near z_0. Thus the only poles of f'/f within C are the zeroes and the poles of f, and these are simple poles with residues equal to the multiplicitly of the zero or the negative of the order of the pole, respectively.

Now let $\{z_i\}$ and $\{p_j\}$ denote the zeroes and poles, respectively, of f within C. Then by the foregoing, and the residue theorem,

$$\begin{aligned}\int_C \frac{f'(z)}{f(z)}\,dz &= 2\pi i\left[\sum_{z_i} \operatorname{Res}_{z_i}\frac{f'(z)}{f(z)} + \sum_{p_j}\operatorname{Res}_{p_j}\frac{f'(z)}{f(z)}\right]\\ &= 2\pi i\left[\sum m_i - \sum n_j\right] = 2\pi i(Z-P),\end{aligned}$$

where m_i denotes the multiplicity of the zero z_i and n_j denotes the order of the pole p_j, and the last equation follows by the definition of Z and P.

The following result can be derived from this after a further study of the geometrical meaning of Rouché's Theorem. We shall give more details on all of this later.

Rouché's Theorem. Suppose that f and g are analytic within and on a simple closed curve C, and that $|f(z)| > |g(z)|$ everywhere on C. Then $f+g$ and f have the same number of zeroes inside C.