Summary:
- We tie down a few loose ends from previous lectures.
- We prove Jordan's lemma and give examples of its application to the evaluation of definite integrals.
- We then give additional examples of finding contours in the complex plane for the evaluation of definite integrals on the real line.

(Goursat, §§41, 44 – 46.)

**31. On zeroes and poles.** Recall that we have defined poles of a definite order and zeroes of a definite order, as follows:

Pole of order $m$ at $a$:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{m} b_n(z-a)^{-n}, \quad b_m \neq 0$$
$$= \frac{\phi(z)}{(z-a)^m}, \quad \phi \text{ analytic and nonzero at } a$$

Zero of order $m$ at $a$:

$$f(z) = \sum_{n=m}^{\infty} a_n(z_a)^n, \quad a_m \neq 0$$
$$= (z-a)^m\phi(z), \quad \phi \text{ analytic at } a.$$

We now claim that function which is analytic except for isolated singularities can have only finitely many poles, and that a nonzero analytic function can have only finitely many zeroes, on any finite region. Thus, let $C$ be a simple closed curve on and within which the function $f$ is analytic, and suppose first that $f$ has infinitely many zeroes at $a_1$, $a_2$, $\cdots$ within the curve $C$. We shall only give the main idea (the details will be given later when we talk about analytic continuation). We need the celebrated *Bolzano-Weierstrass Theorem*:

Let $\{a_1, a_2, \cdots\}$ be an infinite set within a simple closed curve $C$. Then there must be a point $a$ within or on $C$ such that every disk around $a$ contains infinitely many points of this set.

This is proved in courses on analysis, but it is also quite reasonable intuitively since if there infinitely many points in a finite region, surely they cannot all be staying a finite distance away from each other: they must be 'clustering' somewhere.[1] By this theorem, there must be a point $a$ within or on $C$ such that every disk around $a$ contains infinitely many zeroes of $f$; thus any disk around $a$ must contain some point at which $f$ is zero, which means that $f(a)$ must be zero: in other words, $a$ is a zero of $f$. Let us write out the Taylor series of $f$ at $a$:

$$f(z) = \sum_{n=1}^{\infty} \alpha_n(z-a)^n.$$

We claim that all of the coefficients must be zero. Suppose that $\alpha_k \neq 0$ for some $k$. Then we would be able to write

$$f(z) = (z-a)^k\phi(z),$$

where $\phi$ is analytic at $a$ and – crucially – *nonzero* at $a$. Now this would imply that there would be a disk around $a$ on which $\phi$ is still nonzero; but since $(z-a)^k$ is zero only when $z = a$, $f$ would not be zero anywhere on this disk either, contradicting our choice of $a$. Thus all of the coefficients in the Taylor series of $f$ must be zero, which means that $f$ must be identically zero on every disk around $a$ at which it is analytic. Note though that this does not automatically allow us to conclude that it must be identically zero on $C$. The idea to complete the proof – which we shall go over more carefully when we talk about analytic continuation later – is as follows. $f$ must be analytic on some disk around $a$. Now let us take a point near the boundary of this disk; then since $f$ is identically zero near that point, its Taylor series around that point must still be identically zero. Thus $f$ must be identically zero on all disks around this new point on which it is still analytic. We can then continue extending the region until we show that $f$ must actually be identically zero on all of $C$. (Specifically, as we shall see when we talk about analytic continuation, we actually proceed by extending $f$ along a curve to any other point in $C$, which allows us to conclude that $f$ must still be zero at that point, and hence at every point in $C$.)

[1] In other words, there is no way for an infinite group of people to practice social distancing within a grocery store!

Now suppose that the points $a_1$, $a_2$, $\cdots$ were in fact poles. Then as before there would be a point $a$, any disk around which would contain infinitely many of the poles $a_i$. Then clearly $a$ cannot be an isolated singularity of $f$, since any disk around it contains additional singularities of $f$; but $a$ cannot be a regular point either, since $f$ is not analytic on any disk around $a$. Thus $f$ could not be analytic except for isolated singularities, completing the proof in this case.

**32. Cauchy principal value.** Recall that in elementary calculus we give the following definition:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{L_1\to-\infty}\int_{L_1}^{a} f(x)\,dx + \lim_{L_2\to\infty}\int_{a}^{L_2} f(x)\,dx, \tag{1}$$

where the integral on the left exists if and only if the two limits on the right-hand side both exist as finite numbers. Here $a$ is any real number; it is easy to show that the definition does not depend on the choice of $a$, so for convenience we shall take $a = 0$.

Now the careful student may have noted that the integrals we have calculated so far are *not* in the form of a sum of two different limits, but rather of a single limit,

$$\lim_{L\to\infty}\int_{-L}^{L} f(x)\,dx.$$

This limit is called the *Cauchy principal value* of the integral, and we denote it by $\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx$.[2] Now it is easy to see that if the integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists as defined above, then the Cauchy principal value also exists and is equal to it; for in this case

$$\begin{aligned}\lim_{L\to\infty}\int_{-L}^{L} f(x)\,dx &= \lim_{L\to\infty}\int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx\\ &= \lim_{L\to\infty}\int_{-L}^{0} f(x)\,dx + \lim_{L\to\infty}\int_{0}^{L} f(x)\,dx = \lim_{L_1\to-\infty}\int_{L_1}^{0} f(x)\,dx + \lim_{L_2\to\infty}\int_{0}^{L_2} f(x)\,dx\\ &= \int_{-\infty}^{\infty} f(x)\,dx.\end{aligned}$$

The logic, however, does not work in reverse, and for a very simple reason. Note that in going from the first to the second line above we used the fact that if the limit of two quantities exist, then the limit of their sum exists and equals the sum of the limits. It is, however, most definitely *not* true that if the limit of a sum exists, then the limit of the two terms in the sum both exist! (As a simple example, consider $f(x) = 1 - 1/x$ and $g(x) = 1/x$ as $x \to 0$: clearly, $f(x) + g(x) = 1$, and the limit of this exists as $x \to 0$, while neither $f$ nor $g$ has a limit which exists.) Thus the logic cannot be run backwards. To sum up, then: if $\int_{-\infty}^{\infty} f(x)\,dx$ exists, so does $\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx$, and the two must be equal; but the converse is not necessarily true.

There is one case where the converse is true, though: when $f$ is even. In this case, we see that

$$\lim_{L\to\infty}\int_{-L}^{L} f(x)\,dx = 2\lim_{L\to\infty}\int_{0}^{L} f(x)\,dx, \lim_{L\to-\infty}\int_{L}^{\infty} f(x)\,dx = \lim_{L\to\infty}\int_{0}^{L} f(x)\,dx,$$

so that if the Cauchy principal value exists, then so do both of the limits in (1) above, and hence so does the integral $\int_{-\infty}^{\infty} f(x)\,dx$. To summarise, then, we have

$$\text{if the integral } \int_{-\infty}^{\infty} f(x)\,dx \text{ exists, then so does PV} \int_{-\infty}^{\infty} f(x)\,dx, \text{ and the two are equal;}$$
$$\text{if PV} \int_{-\infty}^{\infty} f(x)\,dx \text{ exists and } f \text{ is even, then so does } \int_{-\infty}^{\infty} f(x)\,dx \text{ and the two are equal.}$$

[2] The terminology may be slightly misleading. The Cauchy principal value is something which can be computed independently of the actual integral as defined above: it requires only evaluating the single limit just given. In particular, we do *not* determine the Cauchy principal value by first evaluating the full integral and then doing something to that number!

Another way of looking at the difference between (1) and the principal value is to note that in (1) we have a two-dimensional limit, while the principal value is effectively the directional limit along the line $L_1 = -L_2$. As we learned in multivariable calculus, if the two-dimensional limit of a quantity exists, then so does the limit along any curve – but if all we know is that the limit along one particular line exists, we really do not know anything at all about the full two-dimensional limit, in general. Thus knowing that the Cauchy principal value exists does not, in general, tell us anything about the integral in (1).

It is worth noting that the techniques we have studied so far all amount to calculating the Cauchy principal value rather than the integral as defined in (1). Hence, if we are asked to compute the integral $\int_{-\infty}^{\infty} f(x)\,dx$, in order to show that it equals the Cauchy principal value we must first show that it exists (as it will, as just shown, when $f$ is even, for example).

**33. Jordan's lemma.** Recall that there are two main steps to computing integrals using contours: one, finding a way of 'closing the contour' in such a way that we can calculate the integral of our function over the additional part of the contour (for example, using a semicircle the integral over which goes to zero as its radius goes to infinity); two, evaluating residues. In the previous lecture we saw additional methods for the second step; now we shall prove a result helping us to deal with the first step. First of all, we note that for $x \in [0, \pi/2)$ we have $0 \le \cos x \le 1$, so

$$\frac{d}{dx}\frac{\sin x}{x} = \frac{x\cos x - \sin x}{x^2} = \frac{x - \tan x}{x^2}\cos x \le 0,$$

since $\tan x \ge x$ for $x \in [0, \pi/2)$. This means that the function $\sin x/x$ is decreasing on $[0, \pi/2)$, so its minimum value on $[0, \pi/2]$ is achieved at $x = \pi/2$, and is therefore $(\sin \pi/2)/(\pi/2) = 2/\pi$. Thus for $x \in [0, \pi/2]$ we have $\sin x \ge \frac{2}{\pi}x$.

With this preliminary, we may now prove Jordan's lemma:

*Let $C_R$ denote the semicircle of radius $R$ centred at the origin in the upper half-plane. Let $f$ be a function which is analytic in the upper half-plane on the exterior of some semicircle of radius $R_0$, and such that for every $R > R_0$ there is a constant $M_R$ such that $|f(z)| \le M_R$ on $C_R$, and $M_R \to 0$ as $R \to \infty$. Then $\int_{C_R} f(z)e^{iaz}\,dz \to 0$ as $R \to \infty$ for any positive number $a$.*

To prove this, parameterise $C_R$ by $z(t) = R(\cos t + i\sin t)$, $t \in [0, \pi]$; then

$$e^{iaz} = e^{iaR(\cos t + i\sin t)} = e^{iaR\cos t}e^{-aR\sin t}.$$

Now the first factor has modulus one, while for $t \in [0, \pi/2]$ we have

$$\sin t \ge \frac{2}{\pi}t, \qquad -\sin t \le -\frac{2}{\pi}t, \qquad e^{-aR\sin t} \le e^{-\frac{2aR}{\pi}t};$$

thus

$$\begin{aligned}\left|\int_{C_R} f(z)e^{iaz}\,dz\right| &\le \int_0^{\pi} \left|f(Re^{it})\right| e^{-aR\sin t}R\,dt \le RM_R\int_0^{\pi} e^{-aR\sin t}\,dt = 2RM_R\int_0^{\pi/2} e^{-aR\sin t}\,dt \\ &\le 2RM_R\int_0^{\pi/2} e^{-\frac{2aR}{\pi}t}\,dt = -\frac{M_R\pi}{a}e^{-\frac{2aR}{\pi}t}\Big|_0^{\pi/2} = \frac{M_R\pi}{a}\left(1 - e^{-aR}\right),\end{aligned}$$

which goes to zero as $R \to \infty$, since $M_R$ does and the quantity in parentheses goes to 1. This completes the proof.

We note that the same is true if we replace 'upper' everywhere by 'lower' and require that $a$ be negative: for now we may parameterise $C_R$ by $z(t) = -R(\cos t + i\sin t)$, in the which case, proceeding as before, the integral over $C_R$ of $f(z)e^{iaz}$ can be bounded by

$$2RM_R\int_0^{\pi/2} e^{aR\sin t}\,dt.$$

But now, as before, $aR$ is negative since $a$ is, and the proof proceeds as before with $-aR$ replaced by $aR$ everywhere.

We now give some examples.

EXAMPLES. 1. Evaluate $\int_0^\infty \frac{x\sin x}{1+x^2}\,dx$.

We first note that the integrand is even so that we have

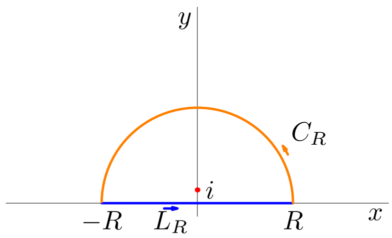
$$\int_0^\infty \frac{x\sin x}{1+x^2}\,dx = 2\int_{-\infty}^\infty \frac{x\sin x}{1+x^2}\,dx,$$

where this last integral will exist exactly when the principal value exists, as shown above. Thus it suffices to compute the principal value of this last integral. We wish to apply Jordan's lemma. Now we have $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$; but if we were to use this formula, it would require us to compute two separate integrals which would need to be closed in different half-planes. That would be possible but would be more work than is necessary. Instead we write

$$\sin x = \operatorname{Im} e^{ix},$$

and note that this allows us to write

$$\int_{-\infty}^\infty \frac{x\sin x}{1+x^2}\,dx = \operatorname{Im}\int_{-\infty}^\infty \frac{xe^{ix}}{1+x^2}\,dx.$$




This looks like the kind of function to which we should be able to apply Jordan's lemma. We only need a bound on $x/(1+x^2)$ on the upper semicircle $C_R$. Now on $C_R$ we have

$$\left|\frac{z}{1+z^2}\right| = \frac{|z|}{|1+z^2|} \geq \frac{R}{R^2-1} = \frac{1/R}{1-R^{-2}},$$

which clearly goes to zero as $R\to\infty$. Thus by Jordan's lemma

$$\int_{C_R} \frac{ze^{iz}}{1+z^2}\,dz \to 0 \quad \text{as} \quad R_{\to}\infty.$$

Since the integrand has only one pole in the upper half-plane, at $z = i$, we may write, by the residue theorem,

$$\begin{aligned}\int_{-\infty}^\infty \frac{xe^{ix}}{1+x^2}\,dx &= \lim_{R\to\infty}\left[-\int_{C_R}\frac{ze^{iz}}{1+z^2}\,dz + 2\pi i \operatorname{Res}_i \frac{ze^{iz}}{z^2+1}\right] \\ &= 2\pi i \operatorname{Res}_i \frac{ze^{iz}}{(z-i)(z+i)} = 2\pi i \frac{ie^{i^2}}{2i} = \frac{2\pi i}{e}.\end{aligned} \tag{2}$$

Thus we have finally

$$\int_0^\infty \frac{x\sin x}{1+x^2}\,dx = \frac{1}{2}\operatorname{Im}\int_{-\infty}^\infty \frac{xe^{ix}}{1+x^2}\,dx = \frac{1}{2}\operatorname{Im}\frac{2\pi i}{e} = \frac{\pi}{e}.$$

It is worth noting that the integral in (2) is a pure imaginary number; this can be traced to the fact that $x\cos x$ is odd, which means that the principal value of its integral over the real line is zero.

2. Evaluate

$$\int_{-\infty}^\infty \frac{\sin x}{x}\,dx. \tag{3}$$

This integral introduces some additional twists to our standard procedure. First of all, by $\sin x/x$ we mean actually the function

$$\begin{cases} \frac{\sin z}{z}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

which by what we have seen on a previous homework assignment is analytic everywhere on the complex plane, and in fact has the Taylor series expansion

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}.$$

If we now think of closing the integral in (3) in the upper half-plane, it would appear that it evaluates to zero since there are no singularities and hence no residue. This would be wrong (and looking at a graph of $\sin x/x$ suggests as much), since as we have noticed before the integral over a semicircle in the upper half-plane of something involving $\sin x$ will not, in general, go to zero since $\sin x$ includes a term $e^{-ix}$. Let us look at this a bit more carefully. Let $L_R$, as usual, denote the line segment from $-R$ to $R$ along the real axis, and $C_R$ denote the upper semicircle of radius $R$ centred at the origin. Then we have by the Cauchy integral theorem

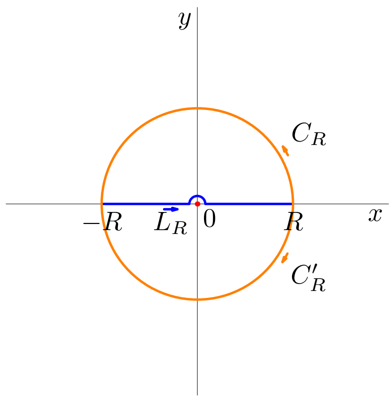
$$0 = \int_{L_R} \frac{\sin x}{x}\, dx + \int_{C_R} \frac{\sin z}{z}\, dz = \int_{L_R} \frac{\sin x}{x}\, dx + \int_{C_R} \frac{e^{iz} - e^{-iz}}{2iz}\, dz.$$

We would like to break this up in such a way that we can close the integral involving $e^{-iz}$ in the lower half-plane. This suggests splitting $\sin x$ up in the first integrand:

$$\int_{L_R} \frac{\sin x}{x}\, dx = \int_{L_R} \frac{e^{ix} - e^{-ix}}{2ix}\, dx.$$

This is perfectly fine, but unfortunately we cannot break this integral up into two separate pieces as it stands since the individual pieces would have a singularity at the origin, which lies on the line $L_R$. (Note that we do need to break this integral up in order to obtain a closed curve with either $C_R$ or $-C_R$ – the lower semicircle – and hence to apply the residue theorem.) But by the Cauchy integral theorem, since $\sin z/z$ is analytic everywhere on the plane, we may replace $L_R$ with any other curve passing from $-R$ to $R$; let us use a contour which goes along the real axis from $-R$ to $-\epsilon$ and $\epsilon$ to $R$, and joins $-\epsilon$ to $\epsilon$ by a small semicircle of radius $\epsilon$ centred at the origin, in the upper half-plane. Denote this contour by $L'_R$. Then we have

$$\int_{L_R} \frac{\sin x}{x}\, dx = \int_{L'_R} \frac{\sin z}{z}\, dz = \int_{L'_R} \frac{e^{iz}}{2iz}\, dz - \int_{L'_R} \frac{e^{-iz}}{2iz}\, dz.$$




We may now evaluate these integrals by closing in the upper and lower half-planes, respectively. Let us look at the first integral. By the residue theorem,

$$\int_{L'_R} \frac{e^{iz}}{2iz}\, dz + \int_{C_R} \frac{e^{iz}}{2iz}\, dz = 0.$$

But now on $C_R$ we clearly have

$$\left|\frac{1}{2iz}\right| = \frac{1}{2}R^{-1} \to 0 \quad \text{as} \quad R \to \infty,$$

so by Jordan's lemma we have

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{2iz}\, dz = 0.$$

Thus

$$\lim_{R\to\infty} \int_{L'_R} \frac{e^{iz}}{2iz}\, dz = 0.$$

The second integral is more interesting. We have by the residue theorem

$$\int_{L'_R} \frac{e^{-iz}}{2iz}\, dz + \int_{-C_R} \frac{e^{-iz}}{2iz}\, dz = 2\pi i \operatorname{Res}_0 \frac{e^{-iz}}{2iz} = \pi,$$

while $|1/(2iz)| = 1/(2R) \to 0$ as $R \to \infty$ shows that the second integral vanishes as $R \to \infty$, by Jordan's lemma. Thus we have

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \text{PV} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \lim_{R\to\infty} \int_{L_R} \frac{\sin x}{x}\, dx = \pi.$$

It is worth noting that the method we used in the previous example – of replacing $\sin x$ by $e^{ix}$ and then taking an imaginary part – does not work directly in this case since the curve $L'_R$ we integrate over does not lie along the real axis, so we cannot simply recover the integral over it of $\sin x/x$ from that of $e^{ix}/x$ by taking an imaginary part.

**34. Another way of closing the contour.** Let us consider, by way of example, another method for closing the contour.

Example. Evaluate the integrals

$$\int_0^{\infty} \sin x^2\, dx, \qquad \int_0^{\infty} \cos x^2\, dx.$$

We do this by considering the integral

$$\int_0^{\infty} e^{iz^2}\, dz.$$

Let $L_R$ in this case denote the line segment from 0 to $R$ along the real axis. We shall close the contour in two different ways. First, though, it is probably worthwhile to consider why the methods we have been using so far do not work in this case. Clearly, if the above integral exists then it will equal half the Cauchy principal value of

$$\int_{-\infty}^{\infty} e^{iz^2}\, dz.$$

Now if we consider the integral from $-R$ to $R$ of $e^{iz^2}$ and then close it with the semicircle $C_R$ of radius $R$ centred at the origin in the upper half-plane, then it would appear initially – as in the previous example – that we would get zero. However, as before, the integral along $C_R$ of $e^{iz^2}$ does not vanish as $R \to \infty$. While not a proof, we may see that this is reasonable by the following computation:

$$\int_{C_R} e^{iz^2}\, dz = \int_0^{\pi} e^{iR^2 e^{2it}}\, iRe^{it}\, dt = \int_0^{\pi} e^{iR^2 \cos 2t} e^{-R^2 \sin 2t} iRe^{it}\, dt;$$

but $\sin 2t < 0$ when $t \in (\pi/2, \pi)$, so that the exponential above goes to infinity with $R$ on that range of $t$. Thus evidently we need to do something else.

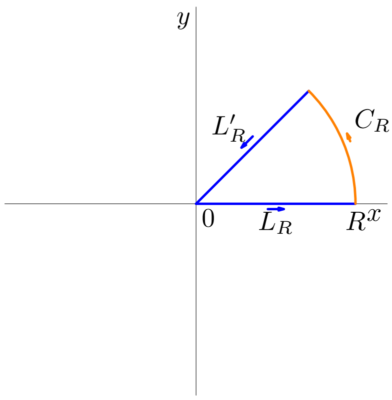
This calculation actually suggests something worth noting: if we were able to restrict $t \in (0, \pi/2)$, then the exponential above would go to zero as $R \to \infty$, and it is possible that the whole integral will also go to zero. Thus let us consider closing the contour with a piece $C'_R$ of the full semicircle $C_R$ together with a line segment back to the origin, i.e., with a pie-wedge shaped contour as in the following figure. The problem now will be how to calculate the integral over the additional line segment $L'_R$. Now the line segment $L'_R$ may be parameterised as $\omega(R-t)$, $t \in [0, R]$ (the $R-t$ is because the line starts on $C_R$ and ends at the origin), where $\omega$ is some complex number of unit modulus. This allows us to write the integral over $L'_R$ as

$$\int_0^R e^{i\omega^2(R-t)^2}\,\omega(-dt) = -\omega \int_0^R e^{i\omega^2 t^2}\,dt.$$

Now if $\omega^2 = i$, then the integrand would become $e^{-t^2}$, and we can compute the integral of $e^{-t^2}$ over the positive real axis by other methods, so it appears that we might be able to use the line $L'_R$ in that case. We shall do this in detail below. Alternatively, if $\omega^2 = -1$, then the integral over $L'_R$ will be $-\omega$ times the conjugate of that over $L_R$, so we may be able to find the integral over $L_R$ in this case also by isolating and solving. We shall not use this method here, but it is very useful for this week's homework assignment. (Hint, hint!)

We shall thus take $\omega$ to satisfy $\omega^2 = i$. This means that we have two choices: $\omega = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ and $\omega = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$. We take the first because to close to the second would require us to use a circle along which we do not have good bounds for $e^{iz^2}$. Thus we let $L'_R$ be the line parameterised by $(R-t)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$, $t \in [0, R]$, and let $C'_R$ denote the segment of the semicircle of radius $R$ centred at the origin from $R$ to $R\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$. Then by the Cauchy integral theorem we have

$$\int_{L_R} e^{iz^2}\,dz + \int_{C'_R} e^{iz^2}\,dz + \int_{L'_R} e^{iz^2}\,dz = 0.$$




We deal with the integral over $C'_R$ first. We have

$$\left|\int_{C'_R} e^{iz^2}\,dz\right| = \left|\int_0^{\pi/4} e^{iR^2e^{2it}} Rie^{it}\,dt\right| \leq R\int_0^{\pi/4} e^{-R^2\sin 2t}\,dt;$$

since $t \in [0, \pi/4]$, we have $2t \in [0, \pi/2]$, so $\sin 2t \geq \frac{2}{\pi}(2t) = \frac{4}{\pi}t$ and the above integral is bounded by

$$R\int_0^{\pi/4} e^{-\frac{4R^2}{\pi}t}\,dt = -\frac{\pi}{4R}e^{-\frac{4R^2}{\pi}t}\Big|_0^{\pi/4} = \frac{\pi}{4R}\left(1 - e^{-R^2}\right),$$

which goes to zero as $R \to \infty$. Thus the integral over $C'_R$ does not contribute anything to the final integral. Now the integral over $L'_R$ is equal to

$$\int_0^R e^{-(R-t)^2}\left[-\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right]dt = -e^{i\pi/4}\int_0^R e^{-t^2}\,dt,$$

which in the limit as $R \to \infty$ becomes

$$-e^{i\pi/4}\frac{1}{2}\sqrt{\pi} = -\frac{1}{2}\sqrt{\frac{\pi}{2}} - i\frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Thus we have finally

$$\int_0^\infty e^{iz^2}\,dz = -\lim_{R\to\infty}\int_{L'_R} e^{iz^2}\,dz = \frac{1}{2}\sqrt{\frac{\pi}{2}} + i\frac{1}{2}\sqrt{\frac{\pi}{2}},$$

so we see that

$$\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$