Summary:
- We give examples of applications of the residue theorem to the evaluation of definite integrals on the real axis.
- We then give theorems for showing that integrals over semicircles go to zero, and provide additional methods for computing residues.

(Goursat, §§44 – 46.)

**28. Evaluation of definite integrals.** Recall the residue theorem from last time: if $f$ is a function which is continuous on a simple closed curve $\gamma$, and analytic inside $\gamma$ except potentially at a finite number of isolated singularities $z_1, \cdots, z_n$, at which it has residues $\beta_1, \cdots, \beta_n$, respectively. Then we have

$$\int_\gamma f(z')\, dz' = 2\pi i \sum_{j=1}^{n} \beta_j.$$

Let us see by way of an example – which could also be done by elementary methods – how this result can be applied to evaluate definite integrals.

EXAMPLE. Consider the integral

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx.$$

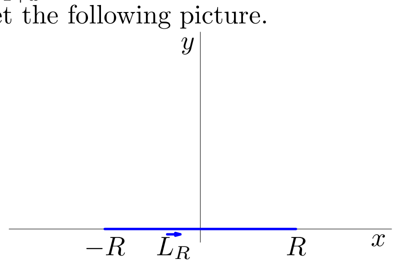
Since this integral converges, it is equal to the limit

$$\lim_{R\to+\infty} \int_{-R}^{R} \frac{1}{1+x^2}\, dx;$$

since $\arctan x$ is an antiderivative of $1/(1+x^2)$, by the fundamental theorem of calculus this integral is equal to

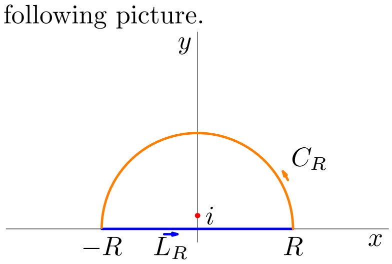
$$\lim_{R\to+\infty} (\arctan R - \arctan(-R)) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Now suppose that we consider $\int_{-R}^{R} \frac{1}{1+x^2}\, dx$ as a contour integral in the complex plane, with the contour taken along the real axis; then we get the following picture.




As it stands this does not seem to have gotten us anywhere. Note though that the integrand here is analytic on the entire plane except for (simple) poles at $\pm i$. Thus if it were possible to somehow *augment* the contour $L_R$ in order to obtain a closed curve (we speak of *closing the contour*), we would be able to apply either the Cauchy integral theorem – if the closed curve did not contain either of the poles – or the residue theorem – otherwise – in order to evaluate the integral over the full closed contour. If, additionally, it were possible somehow to compute the integral over the additional contour, at least in the limit of large $R$, we would then be able to compute our original integral.

In general there are multiple ways of closing the contour; i.e., there are multiple different possible choices for the additional curve to be used to produce a closed contour from the original one. Consider the semicircle $C_R$ in the upper half-plane as in the following picture.



Remember that our technique will only be useful if we have a way of computing $\int_{C_R} 1/(1+z^2)\,dz$; we claim that in fact

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{1+z^2}\,dz = 0.$$

In class we showed this by parameterising the curve $C_R$ and considering the resulting integrand; here we use a slightly simpler method. Recall that, if $f$ is continuous on a simple closed curve $\gamma$, with maximum $M$ on $\gamma$, then

$$\left|\int_\gamma f(z)\,dz\right| \leq M|\gamma|,$$

where $|\gamma|$ is the length of the curve $\gamma$. Now clearly $|C_R| = \pi R$ (since $C_R$ is a *semi*circle of radius $R$); further, if $z$ is any point on $C_R$ then we may write $z = Re^{it}$ for some $t \in [0, \pi]$, and thus

$$\left|\frac{1}{1+z^2}\right| = \left|\frac{1}{1+R^2e^{2it}}\right| = \left|\frac{R^{-2}}{e^{2it}+R^{-2}}\right| = R^{-2}\left|e^{2it}+R^{-2}\right|^{-1}.$$

Now by the triangle inequality we may write

$$\left|e^{2it}+R^{-2}\right| \geq \left|e^{2it}\right| - \left|R^{-2}\right| = 1 - R^{-2},$$

so that when we take the limit $R \to \infty$ we have

$$R^{-2}\left|e^{2it}+R^{-2}\right|^{-1} \leq R^{-2}(1-R^{-2})^{-1} \leq \frac{1}{R^2-1},$$

which goes to zero. Thus

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{1+z^2}\,dz = 0$$

as claimed.

Now for any $R > 1$, the closed curve $L_R + C_R$ will enclose the single pole at $i$. Let us calculate the residue of $1/(1+z^2)$ at $z = i$. We have

$$\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)} = \frac{1/(z+i)}{z-i};$$

if we think of expanding $1/(z+i)$ as

$$\frac{1}{z+i} = \sum_{n=0}^{\infty} c_n(z-i)^n$$

(which we can do since $1/(z+i)$ is analytic near $i$), then

$$\frac{1}{1+z^2} = \frac{c_0 + c_1(z-i) + c_2(z-i)^2 + \cdots}{z-i} = \frac{c_0}{z-i} + c_1 + c_2(z-i) + \cdots,$$

and the residue of $1/(1+z^2)$ is clearly $c_0$. But $c_0 = 1/(z+i)|_{z=i} = 1/(2i)$. Thus at the end of the day we have for all $R > 1$

$$\int_{L_R} \frac{1}{1+z^2}\,dz + \int_{C_R} \frac{1}{1+z^2}\,dz = 2\pi i \cdot \frac{1}{2i} = \pi;$$

and in the limit $R \to \infty$, the integral over $L_R$ goes to $\int_{-\infty}^{+\infty} 1/(1+x^2)\,dx$ while the integral over $C_R$ goes to zero. Thus at the end of the day

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \pi,$$

exactly as we found above.

There are two key points in the above procedure: (i) we have to find a curve $C_R$ (which need not be a semicircle, in general, or any segment of a circular path) which will close the contour $L_R$, and over which we can integrate $f$; (ii) we have to evaluate the residues of $f$ at its singularities inside the closed contour $L_R + C_R$. We shall now give methods for addressing these two points: first, by providing general conditions under which integrals along circular arcs like $C_R$ go to zero as $R \to \infty$; second, by providing additional methods for calculating residues.

**29. When $\int_{C_R} f(z)\,dz \to 0$.** First we have the following fairly straightforward generalisation of the example from the previous section. Suppose that $f$ is a function analytic on the exterior of a circle of radius $R$ for a suitably large $R$ (in other words, if $f$ has any singularities they are not too far from the origin). Suppose that for suitably large $R$ there is a number $M_R$ such that for all $z \in C_R$ we have $|f(z)| < M_R$, and that $\lim_{R\to\infty} RM_R = 0$. Then

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$$

To see this, note that the length of $C_R$ is $\pi R$; thus

$$\left| \int_{C_R} f(z)\,dz \right| \leq \pi R M_R,$$

and as this latter quantity goes to zero by assumption, the integral must also, by the squeeze theorem.

We may apply this to the example in the previous section as follows. Suppose that $z \in C_R$. Then we have

$$|f(z)| = \left| \frac{1}{1+z^2} \right| \geq \frac{1}{|z^2| - 1} = \frac{1}{R^2 - 1},$$

where we have used the triangle inequality as before; since $R/(R^2 - 1) \to 0$ as $R \to \infty$, the above result shows that $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$ as well.

We note in passing that it is actually sufficient to show that if $z \to \infty$ along circles $C_R$, then we must have $\lim_{z\to\infty} zf(z) = 0$. More precisely, what this means is that $zf(z)$ can be made arbitrarily small by taking $z \in C_R$ with $|z| = R$ arbitrarily large. In cases like the foregoing this is easier to apply, since we have

$$\lim_{z\to\infty} \frac{z}{1+z^2} = \lim_{z\to\infty} \frac{z^{-1}}{1+z^{-2}},$$

and since the numerator goes to 0 while the denominator goes to 1, the fraction must go to zero. To be fully rigorous, though, we would have to explain how this kind of a limit – restricting $z$ to lie on a particular family of curves – relates to usual limits, but we shall not do that here. This method can also be used to show easily what we saw in class: suppose that

$$f(z) = \frac{P(z)}{Q(z)},$$

where $P$ and $Q$ are polynomials, and $\deg Q \geq \deg P + 2$. Then we can write

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z(a_0 + \cdots + a_n z^n)}{b_0 + \cdots + b_{n+2} z^{n+2}},$$

where $b_{n+2} \neq 0$ but we may have $a_n = 0$. By dividing numerator and denominator by $z^{n+2}$, this becomes

$$\lim_{z\to\infty} \frac{a_n z^{-1} + a_{n-1} z^{-2} + \cdots + a_0 z^{-n-1}}{b_{n+2} + b_{n+1} z^{-1} + \cdots + b_0 z^{-n-2}} = 0,$$

since the numerator goes to 0 while the denominator goes to $b_{n+2} \neq 0$.

Let us do an example.

EXAMPLE. Compute

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx.$$

We note first that the integrand has poles in the complex plane at $\pm i$, just like the example above. Let us now consider what kind of contour we can use to close the line $L_R$. Now we have

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right).$$

Now suppose that $z = a + ib$; then

$$\cos z = \frac{1}{2}\left(e^{ia-b} + e^{-ia+b}\right),$$

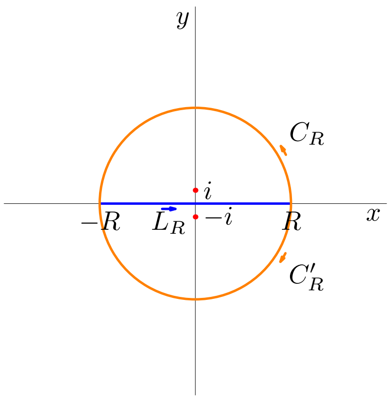
and we see that the first term goes to zero on the upper half-plane ($b > 0$) while the second term goes to infinity exponentially there, and vice versa on the lower half-plane. (We ignore for the moment what happens on the real axis when $b = 0$.) Thus it does not seem that there is any way of closing the contour so as to have $\int_{C_R} f(z)\,dz = 0$ as regardless of whether $C_R$ is in the upper or lower half-plane the integrand will have one term going to infinity.

There are two ways of dealing with this. The more general one is to split the original integral up into two pieces,

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{2(1+x^2)}\,dx, \qquad \int_{-\infty}^{+\infty} \frac{e^{-ix}}{2(1+x^2)}\,dx,$$

and then closing these two integrals in the upper and lower half-plane, respectively; for example, using in turn the curves $C_R$ and $C'_R$ in the following figure. We shall see similar cases to this in the future. For now we use a simpler method. Note that we have also

$$\cos z = \operatorname{Re} e^{iz},$$




so since we are integrating along the real axis, we may write

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx = \int_{-\infty}^{+\infty} \frac{\operatorname{Re} e^{ix}}{1+x^2}\,dx = \operatorname{Re} \int_{-\infty}^{+\infty} \frac{e^{ix}}{1+x^2}\,dx.$$

We try closing this integral in the upper half-plane as described above; thus let $C_R$ denote a semicircle from $R$ to $-R$ in the upper half-plane, as indicated in the above figure. If $z = a + ib \in C_R$, then $b \geq 0$, so we have

$$\left|e^{iz}\right| = \left|e^{ia}e^{-b}\right| = e^{-b} \leq 1,$$

and

$$\left|\frac{e^{iz}}{1+z^2}\right| \leq \frac{1}{|1+z^2|} \leq \frac{1}{R^2-1}$$

as before, and since $\lim_{R\to\infty} R/(R^2-1) = 0$ we have

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{1+z^2}\,dz = 0.$$

Thus we need only calculate the residues of $e^{iz}/(1+z^2)$ in the upper half-plane. Now in the upper half-plane this function is singular only at $z = i$; if we proceed in the same way we did in the previous example (we shall give a general method for this right after this example), we see that the residue is

$$\frac{e^{i\cdot i}}{2i} = \frac{1}{2ei},$$

and finally by the residue theorem and the fact that $\int_{C_R} \frac{e^{iz}}{1+z^2}\,dz \to 0$ as $R \to \infty$, that our original integral is

$$\int_{-\infty}^{+\infty} \frac{e^{iz}}{1+z^2}\,dz = 2\pi i \cdot \frac{1}{2ei} = \frac{\pi}{e}.$$

This is already real, so that we have

$$\int_{-\infty}^{+\infty} \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{e}.$$

**30. Methods for computing residues.** In the previous two examples we have tacitly applied the following result:

Suppose that $f(z)$ has a simple pole at $z = a$. Then

$$\text{Res}_a f(z) = \lim_{z\to a}(z-a)f(z).$$

This is quite simple to see. Since $f$ has a simple pole, there must be a function $\phi(z)$ which is analytic and nonzero at $a$ such that

$$f(z) = \frac{\phi(z)}{z-a}.$$

Then, proceeding as in the two examples above, it is easy to see that $\text{Res}_a f(z) = \phi(a)$; alternatively, we may use the Cauchy integral formula (here $\gamma$ is a small circle around $a$ such that $f$ is analytic on and within $\gamma$):

$$\text{Res}_a f(z) = \frac{1}{2\pi i}\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\int_\gamma \frac{\phi(z)}{z-a}\,dz = \phi(a).$$

But

$$\phi(a) = \lim_{z\to a}\phi(z) = \lim_{z\to a}(z-a)f(z),$$

which establishes our result.

We may extend the above result to poles of higher order. Suppose that $f$ has instead a pole of order $m$ at $z = a$. Then we may write

$$f(z) = \frac{\phi(z)}{(z-a)^m},$$

where as before $\phi$ is analytic and nonzero at $a$; thus (letting as before $\gamma$ denote a small circle around $a$ within and on which $f$ is analytic)

$$\begin{aligned}\text{Res}_a f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{\phi(z)}{(z-a)^m}\,dz = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\phi(z)\bigg|_{z=a}\\ &= \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z).\end{aligned}$$

In other words, to calculate the residue of a function $f$ at a pole of order $m$, we first multiply $f$ by $(z-a)^m$, differentiate $m-1$ times, evaluate at $a$, and divide by $(m-1)!$. (Note that $m \geq 1$, so that $m-1 \geq 0$ and the foregoing makes sense.) Note that this formula reduces to the previous one in the case $m = 1$. Note also that to apply it we must first determine the order of the pole $a$.

In the case that $f(z) = P(z)/Q(z)$, as before, where $P$ and $Q$ have no common factors and $Q$ has no repeated roots, we see that every pole of $f$ will be simple, and the residue at a pole $z_0$ will be

$$\lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}\frac{P(z)}{Q(z)/(z-z_0)} = \frac{P(z_0)}{Q'(z_0)},$$

since $Q(z_0) = 0$ (as $z_0$ is a pole of $f$ and therefore must be a zero of $Q$) and this allows us to write

$$\lim_{z\to z_0}\frac{Q(z)}{z-z_0} = \lim_{z\to 0}\frac{Q(z)-Q(z_0)}{z-z_0} = Q'(z_0).$$

$Q'(z_0) \neq 0$ since by assumption the roots of $Q$ are not repeated.

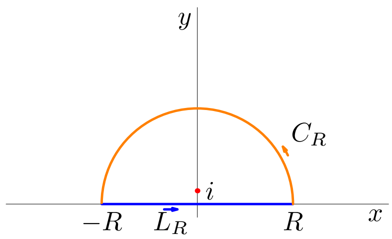
Let us give an example.

EXAMPLES. 1. Evaluate the integral

$$\int_{-\infty}^{+\infty}\frac{1}{(1+x^2)^2}\,dx.$$

We note that the integrand, extended to the complex plane, has poles at $\pm i$, each of order 2. We expect that we can close the contour using a half-circle $C_R$ from $R$ to $-R$ in the upper half-plane, as we have done in the other examples above (see the picture). To see that we can in fact do this, we apply the result from the previous section: for $z \in C_R$, we have

$$|1+z^2| \geq R^2 - 1, \qquad |1+z^2|^2 \geq (R^2-1)^2, \qquad \left|\frac{1}{(1+z^2)^2}\right| \leq \frac{1}{(R^2-1)^2},$$




and since $R/(R^2-1)^2$ clearly goes to zero as $R \to \infty$, we have

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{(1+z^2)^2}\,dz = 0.$$

Thus we need only calculate the residue of $1/(1+z^2)^2$ at $i$. Since $i$ is a pole of order 2 of $1/(1+z^2)^2$, this will be, since $(1+z^2)^2 = (z-i)^2(z+i)^2$,

$$\frac{1}{(2-1)!}\lim_{z\to i}\frac{d^{2-1}}{dz^{2-1}}(z-i)^2\cdot\frac{1}{(1+z^2)^2} = \frac{d}{dz}\frac{1}{(z+i)^2}\bigg|_{z=i} = -\frac{2}{(z+i)^3}\bigg|_{z=i} = -\frac{2}{-8i} = -\frac{1}{4}i,$$

and the integral will be

$$\int_{-\infty}^{+\infty}\frac{1}{(1+x^2)^2}\,dx = 2\pi i\cdot\left(-\frac{1}{4}i\right) = \frac{\pi}{2}.$$

(It is worth pointing out here that it is *always* a good idea to make sure that our final answer makes sense: for example, here we are integrating a real-valued function, so we expect to get a real number as the result; and it is in fact a *positive* real-valued function, so we expect to get a positive real number as the result, as we have. Had we gotten a negative real number, or a complex number with a nonzero imaginary part, it would mean we had made a mistake somewhere earlier which we would need to go back and fix.)

2. Let us evaluate the integral

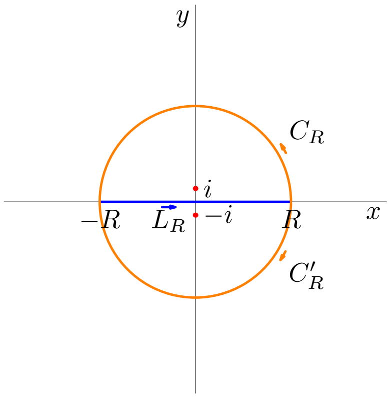
$$\int_{-\infty}^{+\infty}\frac{e^{ikx}}{(1+x^2)^2}\,dx.$$

Here we clearly have the same kind of issue that we had in our example in section 29 above; namely, whether we close the contour in the upper or the lower half-plane will depend on the sign of $k$. Let us first suppose that $k \geq 0$. Then if $z = a + ib$ is in the upper half-plane, so that $b \geq 0$, then as in the example just cited we have

$$e^{ikz} = e^{-kb} e^{ika},$$

which is bounded in absolute value by $e^{-kb} \leq 1$; in other words, the integrand here on the semicircle $C_R$ in the next figure will be bounded by the same quantity as we had for $1/(1+z^2)^2$ in the previous example, and the integral over $C_R$ will go to zero as $R \to \infty$ as there. More carefully, recall that we just showed that on $C_R$

$$\left| \frac{1}{(1+z^2)^2} \right| \leq \frac{1}{(R^2-1)^2};$$




thus when $k \geq 0$ and $z \in C_R$ (so that $z$ is in the upper half-plane)

$$\left| \frac{e^{ikz}}{(1+z^2)^2} \right| \leq \frac{1}{(R^2-1)^2}$$

as well, and since $R/(R^2-1)^2 \to 0$ as $R \to \infty$, we have

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{ikz}}{(1+z^2)^2}\, dz = 0$$

by our general results above. Thus it suffices to calculate the residue of $e^{ikz}/(1+z^2)^2$ at the pole $i$, and since this is still a pole of order 2, its residue is

$$\frac{1}{(2-1)!} \lim_{z\to i} \frac{d^{2-1}}{dz^{2-1}} (z-i)^2 \cdot \frac{e^{ikz}}{(1+z^2)^2} = \frac{d}{dz} \frac{e^{ikz}}{(z+i)^2}\bigg|_{z=i}$$
$$= \frac{ike^{ikz}}{(z+i)^2}\bigg|_{z=i} - 2\,\frac{e^{ikz}}{(z+i)^3}\bigg|_{z=i} = \frac{ike^{-k}}{-4} - 2\frac{e^{-k}}{-8i} = -ie^{-k}\left(\frac{1}{4} + \frac{1}{4}k\right),$$

so that the integral for $k \geq 0$ is

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1+x^2)^2}\, dx = 2\pi i \cdot (-i) e^{-k} \left(\frac{1}{4} + \frac{1}{4}k\right) = \frac{\pi}{2} e^{-k}(k+1).$$

Now suppose that $k \leq 0$, and consider the curve $C'_R$ in the above figure. If $z = a + bi \in C'_R$, then $b \leq 0$, so

$$e^{ikz} = e^{-kb} e^{ika}$$

will still be bounded by 1 in absolute value since $k \leq 0$ and $b \leq 0$ implies that $kb \geq 0$, i.e., $-kb \leq 0$. The exact same logic used above now shows that

$$\lim_{R\to\infty} \int_{C'_R} \frac{e^{ikz}}{(1+z^2)^2}\, dz = 0,$$

and we are left with calculating the residue at $-i$. This is very similar to calculating the residue at $i$; it is equal to

$$\frac{d}{dz}\frac{e^{ikz}}{(z-i)^2}\bigg|_{z=-i} = \frac{ike^{ikz}}{(z-i)^2}\bigg|_{z=-i} - 2\,\frac{e^{ikz}}{(z-i)^3}\bigg|_{z=-i} = \frac{ike^k}{-4} - 2\frac{e^k}{8i} = ie^k\left(\frac{1}{4} - \frac{1}{4}k\right).$$

Before we can determine the value of the integral, though, there is one additional wrinkle we have not yet mentioned: note that the curve $L_R + C_R$ was oriented counterclockwise, as required by the Cauchy integral formula; but $L_R + C'_R$ is oriented *clockwise*, which means that we must put in an extra minus sign when applying the Cauchy integral formula. More carefully, we have

$$\int_{L_R} \frac{e^{ikz}}{(1+z^2)^2}\,dz + \int_{C'_R} \frac{e^{ikz}}{(1+z^2)^2}\,dz = -2\pi i \mathrm{Res}_{-i}\frac{e^{ikz}}{(1+z^2)^2};$$

thus, finally, our integral is

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1+x^2)^2}\,dx = -2\pi i \cdot \left[ie^k\left(\frac{1}{4} - \frac{1}{4}k\right)\right] = \frac{\pi}{2}e^k\,(1-k).$$

Pulling all of this together, then, we have finally that

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{(1+x^2)^2}\,dx = \frac{\pi}{2}e^{-|k|}(1+|k|),$$

since $|k| = k$ when $k \geq 0$ and $|k| = -k$ when $k \leq 0$.

For those who know something about Fourier transforms, it is interesting to note the following about the differentiability of this function. If we expand the exponential out in a Taylor series, we see that (dropping the $\pi/2$ coefficient for convenience)

$$\begin{aligned} e^{-|k|}(1+|k|) &= \left(1 - |k| + \frac{1}{2}|k|^2 - \frac{1}{6}|k|^3 + \cdots\right)(1+|k|) \\ &= 1 - |k|^2 + \frac{1}{2}|k|^2 + \frac{1}{2}|k|^3 - \frac{1}{6}|k|^3 + \cdots. \end{aligned}$$

Now a little thought should convince you that $|k|^n$ has $n$ derivatives everywhere and $n+1$ derivatives everywhere except 0, where the $n+1$th derivative is discontinuous. This shows that $e^{-|k|}(1+|k|)$ has 3 continuous derivatives, which agrees nicely with the fact that the function $1/(1+x^2)^2$ has 3 moments in $L^2$ (i.e., that the integral of the square of $x^n/(1+x^2)^2$ over all of $\mathbf{R}^1$ will be finite for $n \leq 3$).