Summary:

- We give an additional formula for the coefficients in a Laurent series expansion, and discuss how to determine the region of convergence of the series.
- We then discuss poles, essential singularities, and zeros, based on Taylor and Laurent series expansions.
- We define residues, and state the residue theorem.

(Goursat, §§40 – 43.)

24. Laurent series, revisited. Recall from before the term test that if we have a function f, analytic between two circles both centred at a, say C and C', with C' contained inside C, then if we define

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz' \quad (n \geq 0), \qquad b_n = \frac{1}{2\pi i}\int_{C'} (z'-a)^{n-1} f(z')\,dz', \quad (n \geq 1) \tag{1}$$

we have the following series expansion for f:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-a)^n}, \tag{2}$$

and the series on the right-hand side converge on the annular region between C and C'. Now since f is analytic between C and C', as are the quantities $(z'-a)^{-(n+1)}$ and $(z'-a)^{n-1}$ in the definitions of a_n and b_n, the Cauchy integral theorem allows us to evaluate a_n and b_n over *any* simple closed curve γ which lies entirely in the annular region between C and C'. Suppose that we therefore replace C and C' in (1) by γ. Then, noting that we may write

$$b_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{(z'-a)^{-n+1}}\,dz',$$

we see that if we define

$$J_n = \begin{cases} a_n, & n \geq 0, \\ b_{-n}, & n < 0, \end{cases}$$

we see that we have for all $n \in \mathbf{Z}$, positive and negative,

$$J_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{(z'-a)^{n+1}}\,dz', \tag{3}$$

and moreover that we may write the series expansion for f given above as

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} J_n (z-a)^n + \sum_{n=1}^{\infty} J_{-n}(z-a)^{-n} \\ &= \sum_{n=0}^{\infty} J_n (z-a)^n + \sum_{n=-1}^{-\infty} J_n (z-a)^n = \sum_{n=-\infty}^{\infty} J_n (z-a)^n, \end{aligned} \tag{4}$$

where this last sum is essentially defined by the expression on the right-hand side of the first line.[1]

There is no essential difference between the expansions (2) and (4); they are just different ways of expressing the same information. On the other hand, the expression (3) is certainly more symmetric and probably easier to remember than the expression (1). We pay for this in some sense, though, by the fact that the expansion (4) in some sense hides the singular terms, and we must keep in mind that if n is negative the terms $(z-a)^n$ are singular at $z = a$.

25. Region of convergence. There are various ways of determining the region of convergence of a power or Laurent series. Perhaps the most elementary way is to apply the root test we learned in elementary

[1] Note that this is how we define improper integrals of the form $\int_{-\infty}^{+\infty} f(x)\,dx$ in elementary calculus: we pick some point $a \in \mathbf{R}$ and define the integral to be the sum $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx$, much as we may define this last series as $\sum_{n=-\infty}^{-1} J_n(z-a)^n + \sum_{n=0}^{\infty} J_n(z-a)^n$.

calculus. Let us first recall the root test for a series of positive real numbers: if $c_n \geq 0$ for all n, then the series

$$\sum_{n=0}^{\infty} c_n$$

will converge if the quantity $\lim_{n\to\infty} c_n^{1/n} < 1$ and diverge if $\lim_{n\to\infty} c_n^{1/n} > 1$; if $\lim_{n\to\infty} c_n^{1/n} = 1$ then the test is indeterminate and we must do something else. Now suppose that we have a Taylor series

$$\sum_{n=0}^{\infty} a_n (z-a)^n.$$

This series will converge if it is *absolutely convergent*, i.e., if the series of absolute values

$$\sum_{n=0}^{\infty} |a_n (z-a)^n|$$

converges. If we apply the root test to this series, we see that the series is convergent if

$$\lim_{n\to\infty} |a_n (z-a)^n|^{1/n} = |z-a| \lim_{n\to\infty} |a_n|^{1/n} < 1$$

and divergent if

$$\lim_{n\to\infty} |a_n (z-a)^n|^{1/n} = |z-a| \lim_{n\to\infty} |a_n|^{1/n} > 1.$$

Let us now define the quantity R by

$$R^{-1} = \lim_{n\to\infty} |a_n|^{1/n};$$

if this limit is zero we set $R = \infty$, while if the limit is infinite we set $R = 0$. Then from the foregoing we see that the series will converge if

$$|z-a| < R$$

and diverge if

$$|z-a| > R.$$

(If $|z-a| = R$, the root test fails and the series may either converge or diverge.)

Now suppose that we consider instead the Laurent series

$$\sum_{n=1}^{\infty} b_n (z-a)^{-n};$$

by exactly the same logic, if in this case we define R' by

$$R' = \lim_{n\to\infty} |b_n|^{1/n},$$

then we see that the series will converge if

$$|z-a|^{-1} < \frac{1}{R'}$$

and diverge if

$$|z-a|^{-1} > \frac{1}{R'};$$

in other words, it will converge if

$$|z-a| > R'$$

and diverge if

$$|z-a| < R'.$$

(As usual, the test fails when $|z - a| = R'$, meaning that it tells us nothing about the convergence or divergence of the series.) This means that, while Taylor series converge on disks, the singular part of a Laurent series converges instead on the *exterior* of a disk. The full Laurent series, being the sum of a Taylor series and a singular part, will converge on the intersection of one disk with the exterior of another disk, i.e., on an annulus (exactly as we might expect!).[2]

The preceding method will allow us to find the region where any given Taylor or Laurent series converges, assuming that we can calculate the two limits involved. Thus it is useful when the only thing we know is the series itself. On the other hand, if we know the function f to which the series converges, then there is a much simpler method, as follows. Let us consider Laurent series; the same logic applies to Taylor series (which are, after all, just Laurent series without no singular part, i.e., with singular part equal to 0). Suppose that f is a function which is analytic between the circles C' and C, centred at a (with C' inside of C as usual). Then our results with Laurent series show that the Laurent series of f converges to f everywhere on this annulus. Now if f were analytic on a larger annulus, say between circles C_1' and C_1, also centred at a, then its Laurent series on the annulus between C_1' and C_1 would converge between C_1' and C_1; but by the Cauchy integral theorem (for example, recall that we can define the coefficients using any curve γ lying in the annular region) the coefficients of this Laurent series would be the same as those of the original Laurent series. Thus the original Laurent series must also converge on this larger annulus.

If we take this reasoning to its logical conclusion, we see that the Laurent series will converge on the largest annular region on which f is analytic. To be a bit more careful, recall that we need f to be continuous on the boundary curves C and C'; thus the Laurent series will converge on the largest annular region such that f is analytic in the interior and continuous on the boundary. This means that, if we are attempting to find where the Laurent series for a *given* function f is convergent, we need only determine the points where f is not analytic; the Laurent series will then converge on the largest annular region, centred at a, which does not contain any of these points.

(It is worth pointing out that this method does *not* apply to the case where we are simply given a Laurent series and do not know the function to which it converges. This is because it is in general not possible to determine the points where a function is not analytic simply by examining its Laurent series.)

Let us give an example.

EXAMPLE. Let us define a function f by

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} z^{-n}.$$

We will find where these series converge, and then discuss how to use the formulas above to determine its Laurent series. (In a certain sense this second part is quite pointless since a Laurent series is unique, so the series expansion on the right-hand side *is* the Laurent expansion of f around 0. But on the other hand it will give us some practice in the use of the general formulas.) First of all, it can be shown that

$$\lim_{n\to\infty} (n!)^{1/n} = \infty;$$

thus also

$$\lim_{n\to\infty} [(n!)^2]^{1/n} = \infty,$$

and so the first series must converge on the entire complex plane.

Now it can be shown, using L'Hôpital's rule, that

$$\lim_{n\to\infty} n^{1/n} = 1,$$

[2] Note that it is, in principle, quite possible that this intersection might be empty, i.e., that the disk *outside* of which the singular part converges is larger than the disk *inside* of which the analytic part converges. In this case, the series simply does not converge at any point in the complex plane. If we obtained this result by starting from some specific function, it would indicate that the function was not analytic on any annular region centred at a. (This does not, incidentally, mean that the function is never analytic, just that the region on which it is analytic does not contain any annulus around a.)

whence

$$\lim_{n\to\infty}[n^2]^{1/n} = 1,$$

and the quantity R' above will be 1, meaning that the second series converges when

$$|z| > 1.$$

We pause here for a moment to discuss the relation of this last result to our second method for determining the region of convergence of a Laurent series. Let us define

$$g(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} z^{-n}$$

wherever this series converges. Since the series diverges for $|z| < 1$, the function g must have some singularity on the unit circle. Now the series is clearly absolutely convergent on the unit circle itself (since if $|z| = 1$ the absolute value of the terms of the series is simply $1/n^2$, giving a convergent series); thus g does not diverge at any point on the unit circle. Let us look at its derivative. Differentiating term-by-term, we obtain

$$g'(z) = \sum_{n=1}^{\infty} -\frac{z^{-n-1}}{n};$$

thus

$$zg'(z) = \sum_{n=1}^{\infty} -\frac{z^{-n}}{n},$$

so

$$\frac{d}{dz}[zg'(z)] = \sum_{n=1}^{\infty} z^{-n-1} = z^{-2}\sum_{n=0}^{\infty} z^{-n} = z^{-2}\frac{1}{1-z^{-1}} = \frac{1}{z^2-z} = \frac{1}{z(z-1)}.$$

Now this function clearly has a singularity at $z = 1$, which is on the unit circle. While it is not entirely clear how to determine the full function g given this rather peculiar differential operator on g, we can say that if g were analytic at $z = 1$, then so would be $d/dz[zg'(z)]$; since this latter quantity, as just noted, is *not* analytic at $z = 1$, g cannot be analytic there either. (Some further study suggests that g in fact has a branch point at $z = 1$, but we shall not show that here.) This explains why the series cannot converge on a larger annulus.

Proceeding, let us see how to calculate the coefficients in the Laurent series expansion for f using the formula above. (This is very similar to, though more complicated than, the example we did right before the break.) We shall use the formula (3) with $a = 0$, and γ any curve contained in the region $|z| > 1$:

$$J_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{{z'}^{n+1}}\,dz'.$$

Substituting in the series definition of f, we have

$$J_n = \frac{1}{2\pi i}\int_\gamma \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\frac{{z'}^k}{{z'}^{n+1}} + \sum_{k=1}^{\infty}\frac{1}{k^2}\frac{1}{z'^{,k+n+1}}\,dz',$$

so that if we assume we can interchange sum and integral, then we only need to evaluate the integrals

$$\int_\gamma {z'}^{k-n-1}\,dz', \qquad \int_\gamma z'^{-k-n-1}\,dz'.$$

It is sufficient to determine

$$\int_\gamma {z'}^m\,dz'$$

where m is any integer (positive or negative). Now if $m \geq 0$, then the integrand will be analytic everywhere on the plane (as usual, we set $z^0 = 1$ for all z by convention). Now suppose that $m \leq -1$, and write $n = -m - 1 \geq 0$; then by the general Cauchy integral formula

$$\int_\gamma z'^{\,m}\, dz' = \int_\gamma \frac{1}{z'^{\,-m}}\, dz' = \int_\gamma \frac{1}{z'^{\,n+1}}\, dz'$$
$$= 2\pi i \frac{d^n}{dz^n}[1]|_{z=0},$$

which will be $2\pi i$ if $n = 0$ and 0 if $n > 0$, since the derivative of a constant is 0. Pulling all of this together, then, we see that

$$\int_\gamma z'^{\,m}\, dz' = \begin{cases} 0, & m \neq -1 \\ 2\pi i, & m = -1. \end{cases}$$

(This is a very useful formula to keep in mind, in general.) Applying this to the above formula for J_n, we see that for $n \geq 0$

$$J_n = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \int_\gamma z'^{\,k-n-1}\, dz' + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_\gamma z'^{\,-k-n-1}\, dz'$$
$$= \frac{1}{(n!)^2},$$

since the first integral will be nonzero only when $k - n - 1 = -1$, i.e., $k = n$, in the which case it equals $2\pi i$, while the second will be nonzero only when $-k - n - 1 = -1$, i.e., when $k = -n$; since in the second series $k \geq 1$, while $n \geq 0$, this is impossible. Similarly, when $n \leq -1$

$$J_n = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \int_\gamma z'^{\,k-n-1}\, dz' + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_\gamma z'^{\,-k-n-1}\, dz'$$
$$= \frac{1}{(-n)^2},$$

by the foregoing. Thus we may write

$$J_n = \begin{cases} \frac{1}{(n!)^2}, & n \geq 0, \\ \frac{1}{n^2}, & n \leq -1, \end{cases}$$

which is exactly the coefficients in the original series (as advertised).

26. Isolated singularities. We will now apply Laurent series to study the ways in which functions can fail to be analytic at a single point. More precisely, suppose that $a \in \mathbf{C}$, and let f be a function whose domain contains a. If f is analytic at a, then a is called a *regular point* of f; if f is not analytic at a, then a is called a *singular point* of f. Now suppose that a is a singular point, but that f is analytic everywhere else near a; i.e., that there is some $r > 0$ such that f is analytic on the punctured disk

$$\{z \,|\, 0 < |z - a| < r\},$$

i.e., f is analytic everywhere on the disk of radius r centred at a, except of course at a itself. In this case a is called an *isolated singular point* of f. (Note that branch points are *not* isolated singular points.)

Let us see what we can learn about isolated singular points from Laurent series. Suppose that a is an isolated singular point of a function f, and that f is analytic on the punctured disk

$$\{z \,|\, 0 < |z - a| < r\}.$$

Then on any annular region contained in this punctured disk we will have the Laurent series expansion

$$f = \sum_{n=0}^{\infty} a_n (z - a)^n + \sum_{n=1}^{\infty} b_n (z - a)^{-n},$$

where a_n and b_n are given by the formulas (1) above, and moreover do not depend on the particular choice of annular region; thus we may consider this to be an expansion for f on the entire punctured disk (which is, strictly speaking, *not* an annular region in the sense in which we have been using that term). Let us consider the singular part of this expansion, namely

$$\sum_{n=1}^{\infty} b_n (z-a)^{-n}.$$

There are three possibilities: (i) $b_n = 0$ for all n; (ii) $b_n \neq 0$ for only finitely many n; (iii) $b_n \neq 0$ for infinitely many n. In the first case, it can be shown that by defining $f(a) = a_0$, the function f can be made analytic on the whole disk $\{z \,|\, |z-a| < r\}$, so that the 'singularity' is not really a singularity at all; this is called a 'removable singularity'. In the second case we say that the function has a *pole* at a, while in the third case we say that it has an *essential singularity* at a.

Let us give a couple examples.

EXAMPLES. 1. Let $f = 1/\sin z$; then clearly f is analytic everywhere except where $\sin z = 0$, i.e., everywhere except for $z = n\pi$, $n \in \mathbf{Z}$. Each of these points must therefore be an isolated singularity of f. Let us consider the isolated singularity at $z = 0$, and see if we can determine the Laurent series for f there. Now we have

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = z \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}.$$

Now the final series above must converge on the entire complex plane; essentially, this is because it converges at $z = 0$, and other than at that point it is equal to a function analytic on the punctured plane $\{z \,|\, z \neq 0\}$. (If you want to use this fact in your homework solutions, you need to write in a bit more detail!) Let us denote this function by $\phi(z)$, i.e.,

$$\phi(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}.$$

Then since $\phi(0) = 1 \neq 0$, there must be some disk D around 0 on which $\phi \neq 0$. On this disk, then, $1/\phi$ must be analytic, and hence can be expanded as

$$\frac{1}{\phi(z)} = \sum_{k=0}^{\infty} a_k z^k$$

for some set of coefficients a_k, which we could determine by division of series but won't. Thus f can be written as

$$f(z) = \frac{1}{z\phi(z)} = \frac{1}{z} \sum_{k=0}^{\infty} a_k z^k = \frac{a_0}{z} + \sum_{k=0}^{\infty} a_{k+1} z^k.$$

Thus $z = 0$ is a pole for f. We say that it is a *pole of order 1*; we shall define this in general momentarily.

2. Now let us consider the function $f(z) = e^{1/z}$. Clearly, f is analytic everywhere except at $z = 0$, so that 0 is an isolated singular point of f. Now for any complex number z we have

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k;$$

thus for any nonzero complex number z we must have

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k},$$

and this must therefore be the Laurent expansion for f around 0. Since f has infinitely many nonzero coefficients in its singular part, 0 must be an essential singularity for f.

To return to our general treatment, suppose that a is a pole of a function f; this means that a is an isolated singularity of f, and that the singular part of the Laurent expansion of f around a has finitely many nonzero coefficients. Suppose that m is the largest integer for which $b_m \neq 0$, i.e., that $b_m \neq 0$ while $b_k = 0$ for all $k > m$; then we say that a is a pole *of order* m of f. This explains our terminology in the first example above.

In other words, a function f has a pole of order m at a if near a

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \cdots + \frac{b_m}{(z-a)^m},$$

and moreover $b_m \neq 0$.

There is a nice relationship between poles and zeros. Recall that a polynomial $p = a_0 + \cdots + a_n z^n$ is said to have a *zero of order* m at a point a if it is divisible by $(z-a)^m$, i.e., if there is another polynomial q such that $q(a) \neq 0$ and

$$p(z) = (z-a)^m q(z).$$

Now evidently this same definition can be applied to Taylor series. Specifically, suppose that a function f is analytic near a point a; then we may expand it in its Taylor series about a as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

Now f is said to have a *zero of order* m at a if there is an analytic function ϕ near a such that $\phi(a) \neq 0$ and

$$f(z) = (z-a)^m \phi(z).$$

In this case, it is easy to see that the first m terms of the Taylor series for f must all vanish, i.e., that we must have

$$f(z) = a_m (z-a)^m + a_{m+1} (z-a)^{m+1} + \cdots.$$

Now consider $1/f(z)$ near $z = a$; by the foregoing, we have

$$\frac{1}{f(z)} = \frac{1}{(z-a)^m \phi(z)} = \frac{1}{\phi(z)} (z-a)^{-m};$$

now since $\phi(a) \neq 0$, as we saw in Example 1 above, $\phi(z)$ must be nonzero on some disk centred at a; thus $1/\phi(z)$ must be analytic on this disk, say with power series

$$\frac{1}{\phi(z)} = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

and moreover $c_0 = \frac{1}{\phi(a)} \neq 0$. Thus

$$\begin{aligned} \frac{1}{f(z)} &= \frac{1}{(z-a)^m} \sum_{n=0}^{\infty} c_n (z-a)^n \\ &= \sum_{n=0}^{\infty} c_{n+m} (z-a)^n + \frac{c_{m-1}}{z-a} + \cdots + \frac{c_m}{(z-a)^m}, \end{aligned}$$

which shows that $1/f$ must have a pole of order m at $z = a$. This logic works to show the reverse implication also, namely that if a function g has a pole of order m at $z = a$, then $1/g$ must have a zero of order m at $z = a$. Thus, in some sense, poles and zeroes are complementary to each other.

27. Residues. Suppose that a is an isolated singular point of the function f, and let γ be a simple closed curve in the punctured disk about a on which f is analytic, and which moreover encloses the point a. Then we may write the Laurent series for f about a,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}.$$

Thus, assuming that we can interchange sum and integral, we have

$$\int_\gamma f(z')\,dz' = \sum_{n=0}^{\infty} a_n \int_\gamma (z'-a)^n\,dz' + \sum_{n=1}^{\infty} b_n \int_\gamma (z'-a)^{-n}\,dz';$$

by the same logic we used in the examples in the previous section, all of these integrals will vanish except for the one with the power $(z'-a)^{-1}$, and that one will give $2\pi i$. Thus we have

$$\int_\gamma f(z')\,dz' = 2\pi i b_1.$$

The quantity b_1, which is the coefficient of the $(z'-a)^{-1}$ term in the Laurent expansion of f about $z=a$, is called the *residue* of f at the point a.

At present, this logic might seem slightly circular, since the equation above is actually equivalent to the definition of b_1 given above. If we had no other way to find Laurent series than through the definitions of a_n and b_n in terms of integrals, then this would indeed be circular. However, as our examples above have hopefully suggested, there are ways of computing Laurent series that do not involve calculating integrals at all. This means that there are other methods of computing residues. These methods allow us to apply the formula above in meaningful ways.

We may generalise this result as follows to obtain an even more useful one. Suppose that we begin with a curve γ, and that f is analytic everywhere inside γ except at certain isolated singular points $z_1, \cdots, z_n$. Let β_j denote the residue of f around z_j, for $j = 1, \cdots, n$. Then it follows from the generalised Cauchy theorem that we may write

$$\int_\gamma f(z')\,dz' = 2\pi i \sum_{j=1}^{n} \beta_j.$$

This result, known as the *residue theorem*, is extremely useful in the applications we shall make of contour integrals to evaluating definite integrals on the real line.