Summary:

- We derive the Cauchy integral formula from the Cauchy integral theorem for non-simply connected regions.
- We then proceed to show how it may be applied to derive Taylor and Laurent series expansions, and give a simple example.

(Goursat, §§33, 35, 37.)

24. Cauchy integral formula. Suppose that a function f is analytic everywhere inside a simple closed curve C, and continuous on C. Then from our comment at the end of §21 above it follows that the Cauchy integral theorem applies and we have

$$\int_C f(z)\,dz = 0.$$

Now let us fix some point z_0 in the *interior* of the curve C. Then the function

$$\frac{f(z)}{z - z_0}$$

is clearly analytic everywhere inside C except at the point z_0. If we let C' be a small circle centred at z_0 and contained in the interior of C, say with radius $r > 0$, oriented counterclockwise, then by the discussion and result in §23 above we have

$$\int_C \frac{f(z)}{z - z_0}\,dz = \int_{C'} \frac{f(z)}{z - z_0}\,dz;$$

in other words, we are able to replace the (fairly arbitrary and possibly very complicated) curve C by the (presumably much simpler) curve C'. Now we can make C' as small as we like, and the above result will still hold, since $z = z_0$ is the only point inside C at which the integrand $f(z)/(z - z_0)$ is not analytic. Now f is analytic at z_0, so near z_0 we can write as we have before

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z - z_0)(z - z_0),$$

where $\epsilon(z - z_0) \to 0$ as $z \to z_0$. Thus we may write

$$\begin{aligned}\int_{C'} \frac{f(z)}{z - z_0}\,dz &= \int_{C'} \frac{f(z) - f(z_0)}{z - z_0} + \frac{f(z_0)}{z - z_0}\,dz \\ &= \int_{C'} f'(z_0) + \epsilon(z - z_0) + \frac{f(z_0)}{z - z_0}\,dz. \qquad (1)\end{aligned}$$

The integral of $f'(z_0)$ over C' is clearly zero since $f'(z_0)$ is a constant; we shall show in a moment that the integral of $\epsilon(z - z_0)$ over C' must be zero also. Thus we consider the integral

$$\int_{C'} \frac{f(z_0)}{z - z_0}\,dz.$$

Now C' is a circle of radius r centred at z_0, and can be parameterised as

$$z(t) = z_0 + re^{it}, \qquad t \in [0, 2\pi],$$

whence the integral above becomes[1]

$$\int_0^{2\pi} \frac{f(z_0)}{re^{it}} rie^{it}\,dt = \int_0^{2\pi} if(z_0) = 2\pi i f(z_0).$$

[1] Note that this is not really just a 'substitution' as used in elementary calculus; most obviously, substitution in elementary calculus was only shown for integrals of functions of a real variable, and here we are dealing with functions of a complex variable. More substantively, though, the process by which we reduce a contour integral to a definite integral in terms of a parameterisation of the curve follows from the *definition* of the contour integral as we showed above. The formal similarity is however obvious and worth noting as an aid to memory, though it should be borne in mind that the two processes are not identical.

Note that this does not depend on the radius r. Now, finally, consider the integral

$$\int_{C'} \epsilon(z - z_0)\, dz.$$

To evaluate it, note that since $\epsilon(z - z_0) \to 0$ as $z \to z_0$, by taking r sufficiently small we may assume that $|\epsilon(z - z_0)| < 1$ on C'; thus the absolute value of the above integral satisfies

$$\left| \int_{C'} \epsilon(z - z_0)\, dz \right| \leq 2\pi r;$$

thus if we take the limit as $r \to 0$ this integral must vanish. Now if we investigate equation (1), we find that $\int_{C'} \epsilon(z - z_0)\, dz$ is the *only* term in the whole equation which could depend on r; thus *it* can't depend on r either, so since its limit as $r \to 0$ must vanish, it must actually be zero for all r (all r sufficiently small that C' lies entirely inside C, anyway!). Putting all this together, we obtain finally

$$\int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0),$$

or

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz. \tag{2}$$

This is called the *Cauchy integral formula.* Thus the *Cauchy integral theorem* tells us that the integral of an analytic function around a closed curve is 0, while the *Cauchy integral formula* gives us a formula for calculating the *value* of an analytic function inside some curve in terms of an integral around that curve.

Let us expand on this last point for a bit. In equation (2), z_0 is *any* point inside the curve C. Note though that the right-hand side of the equation depends *only* on the values of f *on* the curve C! In other words, what we have here is a formula which will give us the value of a function at any point inside a curve, given only its values on that curve. In the one-variable case, this would be equivalent to saying that the values of a function at the endpoints of an interval determine the function everywhere inside the integral, a claim so patently false as to be silly. For those of you who have seen some partial differential equations, this property should be reminiscent of the solution to boundary-value problems, particularly for Laplace's equation: there, in fact, if one has a Green's function, one can actually produce an integral formula quite reminiscent of (3) for the value of the solution inside a region given only its values on the boundary of the region.[2]

Let us rewrite equation (3) as

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z}\, dz', \tag{3$'$}$$

to emphasise that what we have on the left-hand side is actually a full function rather than a single value. Now it can be shown (see Goursat, §33) that we can differentiate the right-hand side by taking the derivative under the integral sign. In other words, since the point z in (3') must lie *within* C, it cannot lie on C, so that the quantity z' in the integrand is never equal to z and we may therefore write for every z' on C

$$\frac{d}{dz} \frac{1}{z' - z} = \frac{1}{(z' - z)^2},$$

by the power rule and chain rule for differentiating functions of a complex variable. (Note that, while in the integrand we view $1/(z' - z)$ as a function of z', with z fixed, here we view it as a function of z with z' fixed.) Now assuming that we can differentiate under the integral sign, we may write

$$f'(z) = \frac{d}{dz} \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z}\, dz' = \frac{1}{2\pi i} \int_C \frac{f(z')}{(z' - z)^2}\, dz'.$$

[2] Note that there are some connections between these last two sentences. A harmonic function of a single variable would be an f which satisfied the equation $f'' = 0$; the only solutions to this equation are functions $f(x) = ax + b$, where a and b are constants – and a little thought shows that *these* functions actually *do* satisfy the property just stated: in other words, they *are* determined by their values on the endpoints of any interval! The class of harmonic functions on the line, though, is too small to be very interesting.

Assuming that we may again differentiate under the integral sign, we see that the right-hand side also has a derivative and in fact, since

$$\frac{d}{dz}\frac{1}{(z'-z)^2} = \frac{2}{(z'-z)^3},$$

this derivative is

$$\frac{d}{dz}\frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-z)^2}\,dz' = \frac{1}{2\pi i}\int_C \frac{2f(z')}{(z'-z)^3}\,dz'.$$

Continuing in the same way, then, we may evidently write

$$\frac{d^n}{dz^n}\frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\int_C \frac{n!f(z')}{(z'-z)^{n+1}}\,dz'.$$

Since the integral we are differentiating above is equal to $f(z)$, this shows that $f(z)$ has *arbitrarily many* derivatives, as we have often claimed and never actually proved until now. Note that the only assumption we needed to make was that f be analytic on a certain region; we did *not* need to assume that the derivative of f was continuous, or that the real and imaginary parts of f had continuous partial derivatives. These results now follow as a consequence, since the derivative of f must itself have a derivative, and hence must be analytic, hence continuous, showing that the real and imaginary parts of f do indeed have continuous partial derivatives.

To sum up, then, we have, for any nonnegative integer n, the *Cauchy integral formula*

$$f^{(n)}(z) = \frac{1}{2\pi i}\int_C \frac{n!f(z')}{(z'-z)^{n+1}}\,dz'.$$

Let us give a couple examples.

EXAMPLES. If $f(z) = a$ is some constant, then we have

$$a = f(z) = \frac{1}{2\pi i}\int_C \frac{a}{z'-z}\,dz',$$

i.e., that if z is any point inside the simple closed curve C, then $\int_C \frac{1}{z'-z}\,dz' = 2\pi i$; this is a result worth remembering by itself. Now since f is constant, we must have $f'(z) = 0$, and hence $f^{(n)}(z) = 0$ for all $n \geq 1$; the above formula then gives

$$0 = f^{(n)}(z) = \frac{1}{2\pi i}\int_C \frac{a}{(z'-z)^{n+1}}\,dz',$$

which gives

$$\int_C \frac{1}{(z'-z)^{n+1}}\,dz' = 0$$

whenever z' is inside the simple closed curve C and $n \geq 1$. Note that this does *not* follow from the Cauchy integral theorem since the integrand here is *not* analytic within the curve C. Thus we have an extension of the Cauchy integral theorem in this case. Again, this result is worth remembering all by itself.

25. Taylor series. Now that we know that any analytic function must have arbitrary many derivatives, we know that we can formally write out its Taylor expansion

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(z-a)^n, \tag{4}$$

where a is any point in the region on which f is analytic. The existence of the derivatives of f, though, does not prove that this series actually converges to f anywhere except at $z = a$ (where it does trivially since by convention the series above is simply $f(a)$ when $z = a$). Here we shall derive the Taylor expansion by a different method, namely as an application of the Cauchy integral formula. Our exposition closely follows that of Goursat, §35.

Since the series in (4), if it converges anywhere except at $z = a$, must converge on a disk centred at a, let us take our curve C to be a circle of radius R centred at a. Now for any z inside C we have the Cauchy integral formula for f:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z}\, dz'.$$

We shall show how to expand $\frac{1}{z'-z}$ in a power series. We have

$$\frac{1}{z' - z} = \frac{1}{(z' - a) - (z - a)} = \frac{1}{z' - a} \frac{1}{1 - \frac{z-a}{z'-a}}; \tag{5}$$

factoring out $z' - a$ like this is legitimate since here we are only concerned with the expression $1/(z' - z)$ when z' is a point *on* the curve C, and the point a is inside the curve. In fact, in this case, since the curve C is a circle of radius R centred at a, we actually have $|z' - a| = R$. Suppose that $|z - a| = r$; since z also lies inside C, we must have $r < R$. Now we would like to expand the second term in (5) above in a series. We shall augment our treatment in the lecture by providing a careful proof. (Our treatment in the lecture corresponded to taking $N \to \infty$ immediately and dropping the remainder terms, namely those terms coming from w^{N+1} below.) Recall the geometric series

$$\sum_{n=0}^{N} w^n = \frac{1 - w^{N+1}}{1 - w},$$

which is valid for any complex number $w \neq 1$;[3] from this we have

$$\frac{1}{1 - w} = \sum_{n=0}^{N} w^n + \frac{w^{N+1}}{1 - w}.$$

In our case, this gives from (5)

$$\begin{aligned} \frac{1}{z' - z} &= \frac{1}{z' - a} \left[\sum_{n=0}^{N} \left(\frac{z - a}{z' - a} \right)^n + \frac{1}{1 - \frac{z-a}{z'-a}} \left(\frac{z - a}{z' - a} \right)^{N+1} \right] \\ &= \sum_{n=0}^{N} \frac{(z - a)^n}{(z' - a)^{n+1}} + \frac{1}{z' - z} \left(\frac{z - a}{z' - a} \right)^{N+1}. \end{aligned}$$

Substituting this back in to (4), we see that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C \sum_{n=0}^{N} (z - a)^n \frac{f(z')}{(z' - a)^{n+1}} + \frac{f(z')}{z' - z} \left(\frac{z - a}{z' - a} \right)^{N+1} dz' \\ &= \frac{1}{2\pi i} \left[\sum_{n=0}^{N} (z - a)^n \int_C \frac{f(z')}{(z' - a)^{n+1}}\, dz' \right] + \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z} \left(\frac{z - a}{z' - a} \right)^{N+1} dz'. \end{aligned} \tag{6}$$

Let us consider the last term above. Since f is continuous on C, it must be bounded on C; let $M > 0$ be such that $|f(z')| < M$ when z' is on the curve C. Now since $|z' - a| = R$ and $|z - a| = r < R$, we see that $|z' - z| \geq R - r$ (this is just the triangle inequality $|z' - a| \leq |z' - z| + |z - a|$); thus

$$\left| \frac{1}{z' - z} \right| = \frac{1}{|z' - z|} \leq \frac{1}{R - r}.$$

[3] In fact, this formula is valid in any *ring* as long as $1 - w$ is invertible in that ring; i.e., it is a purely algebraic result.

Further,

$$\left|\frac{z-a}{z'-a}\right|^{N+1} = \left(\frac{r}{R}\right)^{N+1}.$$

Thus the absolute value of the second term can be bounded as follows:

$$\left|\frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\left(\frac{z-a}{z'-a}\right)^{N+1} dz'\right| \leq \frac{1}{2\pi}\cdot 2\pi R\cdot M\cdot \frac{1}{R-r}\cdot\left(\frac{r}{R}\right)^{N+1} = \frac{MR}{1-R}\left(\frac{r}{R}\right)^{N+1}.$$

Since $r < R$, this quantity must go to zero in the limit as $N \to \infty$; substituting this into (6) gives

$$\frac{1}{2\pi i}\left[\sum_{n=0}^{\infty}(z-a)^n \int_C \frac{f(z')}{(z'-a)^{n+1}}\, dz'\right] = f(z) - \lim_{N\to\infty}\frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\left(\frac{z-a}{z'-a}\right)^{N+1} dz' = f(z),$$

or to write it out more clearly,

$$f(z) = \sum_{n=0}^{\infty}(z-a)^n \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\, dz'.$$

But by the Cauchy integral formula for $f^(n)$, the integral here is simply $\frac{1}{n!}f^{(n)}(a)$, and we have thus proven the Taylor series expansion for f,

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(z-a)^n,$$

which will be valid on any disk centred at a on which f is analytic. Note that the above argument shows quite rigorously both that the above series converges and that it converges to $f(z)$, given only the general Cauchy integral formula. So if you had never seen a proof that a Taylor series converges to the function it comes from, now you have!

26. Laurent series. It turns out that for many applications it is important to be able to treat functions which have varies kinds of *singularities*, i.e., which fail to be analytic at various points or regions of the plane. While such functions will still clearly have Taylor series expansions on any disk not containing any of these singularities, it turns out to be useful to consider a more general type of expansion which will represent the function on a region surrounding the singularities. These are called *Laurent series.*

Thus suppose that we have a function f which is analytic on an *annulus*; specifically, suppose that C and C' are two circles, centred at a point a, with radii R and R' respectively, where $R > R'$ (so that C' is the inner circle), and both oriented counterclockwise, and that f is analytic on the region between C and C'. We shall extract a series expansion for f from the general Cauchy integral theorem in the same way we found the Cauchy integral formula and then used it to extract the Taylor expansion for f in the previous two sections. Our first step is thus to produce a generalisation of the Cauchy integral formula to the present case. The generalisation is not at all hard. Let z be any point in the annulus between C and C', and let γ be a small circle centred at z and with radius r, oriented counterclockwise and entirely contained in the region between C and C'. Then by the general Cauchy integral theorem in §23, we have

$$\int_C \frac{f(z')}{z'-z}\, dz' = \int_{C'} \frac{f(z')}{z'-z}\, dz' + \int_\gamma \frac{f(z')}{z'-z}\, dz'.$$

Now since γ is entirely contained in the region between C and C', f must be analytic everywhere on and inside γ, which means that by the usual Cauchy integral formula the second integral above is just

$$\int_\gamma \frac{f(z')}{z'-z}\, dz' = 2\pi i f(z),$$

and the above formula gives

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\, dz' - \frac{1}{2\pi i}\int_{C'} \frac{f(z')}{z'-z}\, dz';$$

in other words, we can generalise the Cauchy integral formula to the case of a function analytic *between* two curves if we integrate over both of them with the correct orientation (equivalently, including the correct minus sign). Evidently we could also extend the formula to a situation where a function was analytic on a region with *multiple* holes, but we do not need that here.

Now the first integral above can be treated exactly as before, giving ultimately

$$\sum_{n=0}^{\infty}(z-a)^n \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz',$$

but note that in this case we *cannot* replace the integral with $f^{(n)}(a)/n!$, since f is not known to be analytic at a (f might not even be defined at a, for that matter!). The second integral can be treated by slightly adapting this method. Since in the second integral the point z' lies on C', letting $|z-a| = r$ we have $|z'-a| = R' < r$; thus we may write

$$-\frac{1}{z'-z} = \frac{1}{z-z'} = \frac{1}{(z-a)-(z'-a)} = \frac{1}{z-a}\frac{1}{1-\frac{z'-a}{z-a}};$$

thus we have an analogue to formula (6) but with z' and z interchanged except inside f:

$$\begin{aligned}
-\frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z'-z}\,dz &= \frac{1}{2\pi i}\left[\sum_{n=0}^{N}(z'-a)^n\int_{C'}\frac{f(z')}{(z-a)^{n+1}}\,dz\right] + \frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z-z'}\left(\frac{z'-a}{z-a}\right)^{N+1}\,dz \\
&= \frac{1}{2\pi i}\left[\sum_{n=0}^{N}\frac{1}{(z-a)^{n+1}}\int_{C'} f(z')(z'-a)^n\,dz'\right] + \frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z-z'}\left(\frac{z'-a}{z-a}\right)^{N+1}\,dz'.
\end{aligned}$$

Since we now have, as just noted, $|z'-a| = R' < r = |z-a|$, the argument given above shows that the second integral vanishes in the limit as $N \to \infty$, and we obtain the series expansion

$$-\frac{1}{2\pi i}\int_{C'}\frac{f(z')}{z'-z}\,dz = \sum_{n=0}^{\infty}\frac{1}{(z-a)^{n+1}}\int_{C'}(z'-a)^n f(z')\,dz'.$$

Thus, finally, we find that $f(z)$ can be expressed as the sum of two series:

$$f(z) = \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{C'}\frac{f(z')}{(z'-a)^{n+1}}\,dz' + \sum_{n=0}^{\infty}\frac{1}{(z-a)^{n+1}}\int_{C'}(z'-a)^n f(z')\,dz'.$$

To simplify this a bit, let us make the definitions

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz' \quad (n \ge 0), \qquad b_n = \frac{1}{2\pi i}\int_{C'}(z'-a)^{n-1}f(z')\,dz', \quad (n \ge 1)$$

where in b_1 we have $(z'-a)^0 = 1$ since $z' \neq a$, as z' is on C' and a is inside C'. Then we can write

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n\frac{1}{(z-a)^n};$$

in other words, whereas in the previous section, when f was analytic everywhere inside the circle C and we could write it as a sum of powers of $z-a$, in the case when f is analytic only on an annular region, we must write f as an infinite series of powers of $z-a$ *and* $1/(z-a)$. This is reasonable since $1/(z-a)$ will not be analytic at $z = a$; but note that f may be singular at other points inside C' than just a.

Before ending with an example, it is probably worthwhile to step back a bit and consider what be the importance of the results we have derived in the last three sections. As a concise summary, and for comparison, these are

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz',\\
f(z) &= \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz',\\
f(z) &= \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{C'} \frac{f(z')}{(z'-a)^{n+1}}\,dz' + \sum_{n=0}^{\infty}\frac{1}{(z-a)^{n+1}}\int_{C'}(z'-a)^n f(z')\,dz',
\end{aligned}$$

where f is assumed to be analytic within the arbitrary simple closed curve C in the first line, within the circle C in the second, and between the circles C' and C in the third. All three of these are *representation formulæ*; i.e., they give $f(z)$ as a special type of expression (an integral in the first case, series in the latter two). One of the uses of formulæof this sort is that they give us concrete ways of writing out f, which allow us to perform certain manipulations which would be much harder without them. Another, slightly more abstract, perspective is that these formulægive us a way of breaking f down into other data, which may encode the information we need for a specific problem in a more convenient way than the map $z \mapsto f(z)$ all by itself. For example, if we are only interested in knowing $f(1)$, then the simpler the formula for f the better; but if we are interested in knowing $\int_{C'} f(z)\,dz$, then the simpler the expression for b_1 the better.

On the other hand, these formulæare so general that it will require a fair bit more work before we get to the concrete applications in which they are so powerful. Thus unfortunately we shall have to stop at the vague indications in the previous paragraph for the time being, with a promise to say more about it later.

Let us do an example.

EXAMPLE. Let p be a positive integer, let $a \in \mathbf{C}$, and define the function f on $C\backslash\{a\}$ by

$$f(z) = \frac{1}{(z-a)^p}.$$

Then f is analytic everywhere on the plane except at $z = a$. (This kind of singularity, incidentally, is called a *pole* of order p; we shall study these systematically later.) Thus we expect to be able to expand f as a Laurent series. Actually it is quite obvious that $f(z)$ as given *is* a (single-term) Laurent series, so actually we already know this without any calculation; but let us work out the integrals anyway to see what happens. In this case we may take C and C' to be any circles centred at a, say with radii R and R', where the only condition on these radii is that $R > R'$. We have first of all

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-a)^{n+1}}\,dz' = \frac{1}{2\pi i}\int_C \frac{1}{(z'-a)^{n+p+1}};$$

now $n \geq 0$, while $p \geq 1$, so $n + p \geq 1$ and by the example we did at the end of §24 above we must have $a_n = 0$. Similarly,

$$b_n = \frac{1}{2\pi i}\int_{C'}(z'-a)^{n-1}f(z')\,dz' = \frac{1}{2\pi i}\int_{C'}(z'-a)^{n-1-p}\,dz';$$

if $1 \leq n < p$ (note that if $p = 1$ there will not be any such n, but that doesn't matter) then we must have $n-1-p < -1$, so this integral is zero for the same reason. Now if instead we have $n > p$, then $n-1-p \geq 0$, so the integrand is actually analytic, and by the Cauchy integral theorem we have again $b_n = 0$. The only case left is $n = p$; in this case we have

$$b_p = \frac{1}{2\pi i}\int_{C'}\frac{1}{z'-a}\,dz' = 1,$$

by the first example at the end of §24 above. Thus we can write

$$f(z) = \sum_{n=0}^{\infty}a_n(z-a)^n + \sum_{n=1}^{\infty}b_n(z-a)^{-n},$$

where $a_n = 0$ for all n and $b_n = 0$ except for $n = p$, where $b_p = 1$. The series on the left thus trivially give $\frac{1}{(z'-a)^p}$, as they should.

For those of you who have seen orthogonal bases in vector spaces with an inner product, it is worth noting the formal similarity between the above procedure and that of determining components along the basis vectors in an orthonormal basis. We are not going to make this formal similarity precise, but it is worth noting anyway.