Summary:
- We fill in some holes in the previous exposition.
- We then proceed to give a proof of the Cauchy integral theorem which does not require continuity of the partial derivatives of the real and imaginary parts of the function.
- We show that analytic functions have antiderivatives, at least on simply-connected regions, which are also analytic, and discuss a connection with branch cuts.
- Finally, we discuss an extension of the Cauchy integral theorem to regions which are not simply connected.

(Goursat, §§28 – 31)

20. A few points from previous material. Recall that we have shown that, if m is a positive integer, then the *power rule* for differentiation on the real line applies also to derivatives in the complex plane:

$$\frac{d}{dz}z^m = mz^{m-1}.$$

The same result holds true for *any* complex exponent m, as long as we interpret the left- and right-hand sides appropriately. To see this, recall that if m is any complex number, we define the exponential z^m by

$$z^m = e^{m\operatorname{Log} z},$$

where $\operatorname{Log} z$ represents the full multivalued complex logarithm of the complex number z. As we discussed when we first gave this definition, the right-hand side is multivalued since Log is. Suppose now that we take a particular branch of Log, say by requiring the angle to lie between $(\theta_0, \theta_0 + 2\pi)$ for some $\theta_0 \in \mathbf{R}$.[1] For this particular branch, as in general,

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z},$$

and by the chain rule we have

$$\frac{d}{dz}z^m = \frac{d}{dz}e^{m\operatorname{Log} z} = e^{m\operatorname{Log} z}\frac{d}{dz}m\operatorname{Log} z = me^{m\operatorname{Log} z}\frac{1}{z} = me^{m\operatorname{Log} z - \operatorname{Log} z} = me^{(m-1)\operatorname{Log} z} = mz^{m-1},$$

where z^{m-1} is taken using the same branch of Log as z^m.[2] Thus we do indeed have

$$\frac{d}{dz}z^m = mz^{m-1},$$

as long as the powers on both sides are computed using the same branch of the logarithm.

Since the functions $z \mapsto z^m$, where m is any nonzero integer, are all single-valued, the result above holds without any conditions for (nonzero) integer exponents. It holds for $m = 0$ if we define z^0 to be 1 everywhere, including at 0. (Recall that 0^0 is not defined.)

We now wish to point out another version of the chain rule involving complex numbers. Recall that, if f and g are two complex-valued functions of a complex variable, both of which are analytic, then $f \circ g$ is also analytic where it is defined, and we have

$$\frac{d}{dz}(f \circ g) = f'(g(z))g'(z).$$

Now suppose that f is an analytic function of a complex variable, and that $\gamma : [a, b] \to \mathbf{C}$ is a smooth curve. Then we have also

$$\frac{d}{dt}(f \circ \gamma) = f'(\gamma(t))\gamma'(t).$$

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[1] It is worth noting here that, although we specify a branch cut for Log and for the root functions by specifying an interval for the angle θ, a branch cut is a cut of the *entire plane*, not just the unit circle.

[2] Note that it is *always* valid to write $z = e^{\operatorname{Log} z}$ and $\frac{1}{z} = e^{-\operatorname{Log} z}$, regardless of the branch of Log we are using (or even if we are not taking a branch at all). The first follows from the definition of Log as the inverse function of exp, and the second follows from the first by laws of exponents.

This can be shown in the same way that we showed the original chain rule above; briefly, we may write

$$f(\gamma(t+h)) = f(\gamma(t) + \gamma'(t)h + o(h)) = f(\gamma(t)) + f'(\gamma(t))(\gamma'(t)h + o(h)) + o(\gamma'(t)h + o(h)),$$

so if we are willing to accept that $o(\gamma'(t)h + o(h))$ is also $o(h)$, this becomes

$$f(\gamma(t+h)) = f(\gamma(t)) + f'(\gamma(t))\gamma'(t)h + o(h),$$

from which the result follows by computing the difference quotient and taking a limit. (Here, again, by $o(h)$ we mean any function – of a real variable in this case – which satisfies $\lim_{h\to 0} o(h)/h = 0$.)

(It is worth noting the difference between these two chain rules. In the first one, both f and g were functions of a *complex* variable, while in the second one f is a function of a complex variable but γ is a function only of a *real* variable. We have been told many times – and will shortly begin to see for ourselves! – that the requirement that a function of a complex variable have a derivative is far more restrictive than the requirement that a function of a real variable have a derivative: note that the difference is between the *domains*, and *not* the ranges. In other words, the difference is between a function *defined* on the complex numbers, and a function *defined* on the real numbers, and not a function taking values in the complex numbers and a function taking values in the real numbers.)

21. The Cauchy integral theorem, full proof. Recall that in section 19 above we showed that, if f is an analytic function on a simply-connected region, and C is any simple (non self-intersecting) closed curve contained in that region, then *if f has continuous first-order partial derivatives on the region,*

$$\int_C f(z)\,dz = 0.$$

We will now show that this result holds *without* the assumption of continuous first-order partial derivatives, which we will actually be able ultimately (next week) to derive as a consequence. Our treatment follows very closely that given in Goursat, §28.

Thus, let f be an analytic function on some region, and let C be any simple closed curve in that region such that f is analytic everywhere on the interior of C. Let U denote the region bounded by C, which is necessarily simply-connected; then by assumption f is analytic on U and on C. Now suppose that we subdivide U into squares and partial squares by drawing a square grid across it (see Figure 13 in Goursat for an example of what we mean by this). We let γ_k denote the boundary curve – oriented counterclockwise – of the kth full square, and γ_j' denote the boundary curve – again oriented counterclockwise – of the jth partial square. Then we claim that

$$\sum_k \int_{\gamma_k} f(z)\,dz + \sum_j \int_{\gamma_j'} f(z)\,dz = \int_C f(z)\,dz.$$

This is clear after a bit of thought, since the sides of the grid squares appear exactly twice, and in opposite directions, in the sum of integrals on the left, and hence cancel, meaning that we are left only with the integral around the boundary curve, i.e., the right-hand side.

We now claim that each of the integrals in the above sums is small. To see this, note that since the functions $z \mapsto a$ and $z \mapsto a(z-z_0)$ are analytic with continuous partial derivatives, the result from section 19 can be applied to show that around any closed curve they integrate to zero. (Another way of showing this, without applying Green's theorem – which we did in section 19 – is outlined in section 28 of Goursat.) Now consider $\int_{\gamma_k} f(z)\,dz$, and let z_0 be some point either inside or on γ_k. Then since f is analytic on U, we can write, for any point z on γ_k,

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \epsilon(z-z_0)(z-z_0),$$

where $\epsilon(z-z_0) \to 0$ as $z \to z - z_0$. (In o notation, $\epsilon(z-z_0) = o(z-z_0)/(z-z_0)$, but we stick with this notation here for consistency with the lecture.) Now the functions $z \mapsto az$ and $z \mapsto a(z-z_1)$, where $a, z_1 \in \mathbf{C}$ are any two constant complex numbers, are both analytic with continuous first-order partials (this is entirely trivial!); thus the result from Section 19 shows that both integrate to zero around any simple closed curve. Hence we may write

$$\int_{\gamma_k} f(z)\,dz = \int_{\gamma_k} f(z_0) + f'(z_0)(z-z_0) + \epsilon(z-z_0)(z-z_0)\,dz = \int_{\gamma_k} \epsilon(z-z_0)(z-z_0)\,dz,$$

and similarly, letting z_0' denote some point within or on γ_j', and $\epsilon'(z-z_0')$ the corresponding function analogous to $\epsilon(z-z_0)$,

$$\int_{\gamma_j'} f(z)\,dz = \int_{\gamma_j'} \epsilon'(z-z_0')(z-z_0')\,dz.$$

Now recall that, if we have a curve γ of length ℓ and an analytic function f which is bounded by M on γ, then we have the bound

$$\left|\int_{\gamma} f(z)\,dz\right| \le \ell M.$$

Let us apply this to the two integrals above. Suppose that $\epsilon(z-z_0) \le \eta$ on γ_k, and that γ_k is a square with side lengths ℓ_k; then the total length of γ_k is $4\ell_k$, and moreover the function $z - z_0$ on γ_k is bounded by $\ell_k\sqrt{2}$ (since this is the length of a diagonal of γ_k and that is the farthest apart any two points can be on a square). Thus we may write

$$\left|\int_{\gamma_k} f(z)\,dz\right| \le 4\ell_k \cdot \eta\ell_k\sqrt{2} = 4\sqrt{2}\ell_k^2\eta = 4\sqrt{2}A_k\eta,$$

where $A_k = \ell_k^2$ is the area enclosed by γ_k. Similarly, suppose that $\epsilon'(z-z_0')$ is bounded by some number η' on γ_j'. Now γ_j' consists of parts of four sides of a square, together with some portion of C; thus, if we let ℓ_j' denote its side length and λ_j the length of that portion of C, then the length of γ_j' is bounded by $4\ell_j' + \lambda_j$. (This may be a very bad upper bound, since we may only have a small portion of the square sides, but the point is that it *is* an upper bound, and as we shall see later, it is a sufficiently good upper bound.) Now because we have decomposed the region U along a square grid, the region enclosed by γ_j' is a portion of a square, i.e., it is a region entirely contained in one of these squares; thus as before the function $z - z_0'$ on γ_j' is bounded by $\ell_j'\sqrt{2}$ and we may write

$$\left|\int_{\gamma_j'} f(z)\,dz\right| \le (4\ell_j' + \lambda_j)\cdot\eta'\ell_j'\sqrt{2} = (4A_j' + \ell_j'\lambda_j)\sqrt{2}\eta'.$$

Now we come to a technical point which is addressed in §29 of Goursat but which we shall just touch on without giving a formal proof. We know that as $z \to z_0$, $\epsilon(z-z_0) \to 0$, and similarly that $\epsilon'(z-z_0') \to 0$ as $z \to z_0'$. Similar relations will be true in all of the other squares and partial squares into which we have subdivided U.[3] This means that, by taking each individual square small, we can make the quantities η and η' small. We claim that by taking the entire *grid* arbitrarily fine, i.e., to have squares and partial squares which are arbitrarily small, *all* of the functions $\epsilon(z-z_0)$ and $\epsilon'(z-z_0)$, for *all* indices k and j (respectively), can *simultaneously* be made arbitrarily small. This does not automatically follow from the foregoing, but as it does seem reasonable, and the proof is slightly technical, we shall assume its truth and see how it can be used to derive the result. (As mentioned, an explanation of this result is given in §29 of Goursat for those who are interested.) Thus we assume that, for any $\eta_0 > 0$, by taking the grid sufficiently fine, we may assume that for all k and j, we may take $\eta, \eta' < \eta_0$. Now consider such a sufficiently fine grid, and let L be the side length of the squares in the grid; then we may write

$$\left|\sum_k \int_{\gamma_k} f(z)\,dz\right| \le \sum_k \left|\int_{\gamma_k} f(z)\,dz\right| \le 4\sqrt{2}\eta_0 \sum_k A_k \le 4\sqrt{2}\eta_0 A,$$

[3] Note that the points z_0 and z_0' actually depend on the indices k and j, respectively, but we have chosen not to indicate this in our notation just for simplicity.

where A denotes the area of some circle completely containing U, and such that all squares in the grid which intersect U are completely contained in that circle; similarly, letting λ denote the length of the curve C,

$$\left|\sum_j \int_{\gamma'_j} f(z)\,dz\right| \le \sum_j \left|\int_{\gamma'_j} f(z)\,dz\right| \le \sqrt{2}\left[4\sum_j A'_j + L\sum_j \lambda_j\right]\eta_0$$

$$\le \sqrt{2}(4A + L\lambda)\eta_0.$$

Thus, finally, we have

$$\left|\int_C f(z)\,dz\right| \le (4\sqrt{2}A + 4\sqrt{2}A + \sqrt{2}L\lambda)\eta_0 = (8\sqrt{2}A + L\sqrt{2}\lambda)\eta_0,$$

where η_0 is an arbitrary positive number. Now if we take any grid *finer* than the one we just considered, clearly L will decrease, while we can use the same A as before; in other words, if $\eta'_0 < \eta_0$ and we consider any grid fine enough to have $\eta, \eta' < \eta'_0$, we may still write

$$\left|\int_C f(z)\,dz\right| \le (8\sqrt{2}A + L\sqrt{2}\lambda)\eta'_0,$$

where A and L have the same values as they did before. By taking η'_0 arbitrarily small, we see that the left-hand side must be arbitrarily small; since it does not depend on the grid, or η'_0, it must actually be zero. This proves that

$$\int_C f(z)\,dz = 0,$$

as claimed.

In the above we have assumed that the function f was defined and analytic on a larger region completely containing the curve C and its interior. It turns out that one only need assume f to be analytic on the interior of C and *continuous* up to the boundary; a brief discussion of this is given in the footnote in Goursat, pp. 48 – 49 (of the typescript; p. 71 of the original).

22. Antiderivatives and branch cuts. Recall that in multivariable calculus we learned that a vector field $\mathbf{F}$ which is *conservative*, in the sense that its integral around any closed curve is zero, has a *potential function*, i.e., that there is a function f such that $\mathbf{F} = \nabla f$. Moreover, f can be constructed as

$$f(x, y) = \int_{(x_0,y_0)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r},$$

where (x_0, y_0) is any fixed point, and the line integral does not depend on the choice of curve from (x_0, y_0) to (x, y) since $\mathbf{F}$ is conservative. We now show a similar result in the case of analytic functions of a complex variable, though as usual the import is quite a bit deeper.

Thus suppose that f is a function analytic on a simply-connected region, pick some point z_0 in that region, and define a function

$$F(z) = \int_{z_0}^{z} f(z')\,dz'.$$

Let us see in what sense this formula defines a function. Recall that a function consists of three things: a domain, a range, and a rule giving an element of the range for any element of the domain. Here the domain can clearly be taken to be the simply-connected region on which f is analytic, and as usual we don't really worry about the range (F will certainly be in $\mathbf{C}$, at any rate). Thus we only need to consider in what sense the function above defines a rule which gives a complex number given any complex number in its domain. In order to evaluate the integral, we need to choose a particular path γ from z_0 to z. Suppose that γ_1 and γ_2 are two distinct paths from z_0 to z. If γ_1 and γ_2 have no intersection points other than their endpoints z_0 and z, then by running γ_1 forwards and γ_2 backwards we obtain a simple closed curve; if we call it γ, then we have

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz;$$

but by the Cauchy integral theorem, the left-hand side is zero, so that $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ and the integral evidently does not depend on the choice of curve in this case. It can be shown that this holds true even if the two curves have other intersection points; thus in the situation we are considering here, $F(z)$ depends only on the endpoints z and z_0, and not on the curve chosen from z_0 to z. It therefore does indeed give a single-valued function on the region.

Let us see whether we can compute its derivative. Thus we consider the quotient $[F(z+h)-F(z)]/h$. Now by choosing the curve used to calculate $F(z+h)$ so that it passes through z, we may write

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_z^{z+h} f(z')\,dz'.$$

Now we note that $\int_z^{z+h} dz' = h$, just as in elementary calculus on the real line (this can be shown by parameterising the line from z to $z+h$, for example); thus this last expression is equal to

$$\frac{1}{h}\int_z^{z+h} f(z') - f(z)\,dz'.$$

But now if h is very small, $f(z') - f(z)$ will be very small for all points on the straight line from z to $z+h$, which means that also $|f(z') - f(z)|$ will also be very small there; if η is any upper bound on this quantity, then we may write

$$\left|\frac{1}{h}\int_z^{z+h} f(z') - f(z)\,dz'\right| \leq \frac{1}{|h|}|h|\eta = \eta,$$

which means that by taking h sufficiently small, the above quantity must be less than η. But if we unravel everything, this means that the limit

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} - f(z)$$

must be zero, which means that F is analytic and $F'(z) = f(z)$, as we might have expected.

It is worth noting that, if z_1 and z_2 are any two complex numbers in the region above, then by taking the curve from z_0 to z_2 to pass through z_1, we may write

$$\int_{z_1}^{z_2} f(z')\,dz' = \int_{z_0}^{z_2} f(z')\,dz' - \int_{z_0}^{z_1} f(z')\,dz' = F(z_2) - F(z_1),$$

which shows that the fundamental theorem of calculus is true in this case as well.

Let us now consider what could have gone wrong if the region on which f was known to be analytic had not been simply connected. For the kinds of regions we are interested in here (essentially, open sets in the plane), the notion of 'simply connected' is a *global* notion, in the sense that it is in general a property of the entire region, not just some portion of the region. Alternatively, any region is *locally* simply connected, since if we consider any point in the region, there is certainly a small disk around that point contained in the region, and that disk will be simply connected. On such a disk, the above logic goes through, and thus we see that, at least near any given point, we can still construct an antiderivative of f in exactly the same fashion as above. What goes wrong is when we try to push this construction further away from the point. Thus suppose for example that f is analytic everywhere except at some point ζ_0, and let $z_0 \neq \zeta_0$; then near z_0 the function

$$F(z) = \int_{z_0}^{z} f(z')\,dz'$$

will be well-defined and independent of the curve connecting z_0 and z, and will give an antiderivative of f. But now consider trying to determine this function everywhere on some circle starting at z_0 which encloses the point ζ_0. At z_0 we have $F(z_0) = 0$ by definition. But when we traverse this circle around ζ_0, as we come back close to z_0, $F(z)$ may not be small, since there is no guarantee that the integral around the entire curve will vanish. This means that the limit of $F(z)$ may not equal 0 as $z \to z_0$ along this direction, and hence that it may not be possible to find a single-valued continuous antiderivative of f everywhere on the region. This is, in fact, a generalisation of what we have seen goes wrong when we consider the logarithm: since $\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}$, $\operatorname{Log} z$ is an antiderivative of $\frac{1}{z}$, and as we try to take its value along some closed curve containing the origin, we know that we run into problems of discontinuity or multivaluedness exactly like those just discussed. One solution to this problem in the general case is to use the solution we used for the logarithm, and take a branch cut starting at ζ_0 and going to infinity; the resulting region will be simply connected, and thus on it we may define a single-valued, continuous antiderivative using the above formula.

The notions above of starting out with an analytic function only defined on a small disk and attempting to extend it further are related to notions of *analytic continuation* which we shall discuss later on in the course.

23. An extension of Cauchy's integral theorem to non-simply connected regions. It turns out that there is a way of extending Cauchy's integral theorem to non-simply connected regions, in quite the same way one extends Green's theorem to such regions, which will be important to our derivation of the Cauchy integral formula and is also noteworthy in its own right. Suppose for definiteness that a function f is analytic everywhere on a region except at two holes (these could be two isolated points, or larger holes), and consider $\int_C f(z)\,dz$, where C is some simple closed curve in this region. As long as C does not enclose either of the holes, this integral will still vanish by the Cauchy integral theorem. Now if C contains just one of the holes, we may shrink it down to either the boundary curve of the hole (if the hole is itself a region) or to an arbitrarily small circle around the hole (if the hole is a point), and the integral of f over this new curve, call it C', will be equal to that of f over C: to see this, think of taking a point on C and joining it to some point on C' by a straight line; if we break this straight line open slightly, and pull the two edges apart, we will get a simple closed curve which does not enclose any singularities of f, and the integral over this curve will therefore vanish; but in the limit as the two lines come together, the integral over this curve is just

$$\int_C f(z')\,dz' - \int_{C'} f(z')\,dz',$$

assuming that we orient both C and C' counterclockwise. Thus these two integrals must be equal, as claimed.

In the case that C is a curve enclosing both holes, we may do something similar except that we will find

$$\int_C f(z')\,dz' = \int_{C_1'} f(z')\,dz' + \int_{C_2'} f(z')\,dz',$$

where C_1' and C_2' are curves enclosing the two holes, as described above. Here we are still assuming that all three curves are oriented counterclockwise. If we instead orient C_1' and C_2' clockwise, and call the resulting curves C_1 and C_2, then the above result becomes

$$\int_C f(z')\,dz' + \int_{C_1} f(z')\,dz' + \int_{C_2} f(z')\,dz' = 0,$$

i.e., the integral of f is still zero as long as we include curves around the singularities of f as well.