Summary:
- We outline a particular application of conformal maps.
- We then define and investigate integrals of complex functions over curves in the complex plane.

(Goursat, §§24, 25 – 26, 32.)

17. Application of conformal maps to harmonic functions. In fields as varied as electrostatics, heat flow, and fluid mechanics (and probably others) one is often interested in solving problems of the following form: we are given a particular region U in the plane[1] with boundary curve C, and some particular function g on the boundary curve C, and we wish to find a function P on U which satisfies

$$\Delta P = 0 \,\text{on}\, U, \qquad P|_C = g.$$

This problem in full generality is a topic for a course in partial differential equations, but there are specific cases which can be treated by using complex variable techniques to replace the region U by another one for which the problem is more tractable. Let us see how this works. (Here we shall simply outline the idea; we shall go over it in more detail later on in the course. Thus what follows is meant to be more of a conceptual introduction than a careful exposition.) Suppose that we have another region U' with boundary curve C' and a conformal map $f : U' \to U$ which maps C' onto C and is also analytic with an analytic inverse $f^{-1} : U \to U'$ (I accidentally forgot about this restriction in the lecture; there are probably ways of treating the problem without it, but we shall leave a detailed discussion of the matter for another time). Then we may consider instead the problem

$$\Delta P' = 0 \,\text{on}\, U', \qquad P'|_{C'} = g \circ f.$$

For a suitable choice of U' and f, this problem may be easier to solve than the original one. Suppose that we are able to find a solution to this modified problem. Then we claim that $P' \circ f^{-1}$ is a solution to the original problem. To see this, let $z_0 = x_0 + iy_0 \in U'$ be some particular point and define

$$Q' = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P'}{\partial y}\, dx + \frac{\partial P'}{\partial x}\, dy,$$

i.e., the conjugate harmonic function to P'; then the function

$$F(x+iy) = P'(x,y) + iQ'(x,y)$$

will be analytic, at least on any simply connected subset of U' containing (x_0, y_0). Since f^{-1} is also analytic, this means that $F \circ f^{-1}$ is also analytic, and hence that its real part

$$P' \circ f^{-1}$$

is harmonic.[2] Further, on C

$$\left(P' \circ f^{-1}\right)\big|_C = P'|_{C'} \circ f^{-1} = g \circ f \circ f^{-1} = g,$$

so that the boundary condition is satisfied as well. Thus $P' \circ f^{-1}$ is indeed a solution to the original problem, as claimed.

(In the lecture I actually showed the *opposite* implication, namely that if P is a solution to the original problem, then $P \circ f$ is a solution to the modified problem. The argument is identical to that here, replacing f^{-1} by f and P' by P as appropriate.)

---

[1] While not strictly necessary, we can assume that U is simply connected below to avoid some technical complications which are not important at the moment.

[2] Note that we can afford to be vague about the region here since the property of being analytic and – especially – harmonic is really a pointwise property; or if we want to be a bit more careful, it is a property we only need to consider on small disks, which are always simply connected.

Appendix I. Formal definition of simple connectedness. Informally, a region which is simply connected is one which has no 'holes'. As mentioned in lecture, if the region is given to us pictorially this is about all we could go on to determine whether it is simply connected. More carefully, though, a region is simply connected if any closed curve can be continuously shrunk to a point within the region. But what do we mean by 'continuously shrunk to a point within the region'?

The precise definition of simply connected, which is valid in any topological space, is as follows. A set U is said to be *simply connected* if for any continuous closed curve $\gamma : [0,1] \to U$ (i.e., γ is continuous and satisfies $\gamma(0) = \gamma(1)$) there is a continuous map $F : [0,1] \times [0,1] \to U$ satisfying the following conditions:

$$\begin{aligned} &F(\cdot, s) : [0,1] \to U \text{ is a closed curve in } U \text{ for each } s \in [0,1] \\ &F(t,0) = \gamma(t) \text{ for all } t \in [0,1] \\ &F(t,1) = u_0 \text{ for all } t \in [0,1] \end{aligned}$$

where $F(\cdot, s)$ means the map $t \mapsto F(t,s)$, and $u_0 \in U$ is some point. If we unwrap this definition a bit, what it means is that $F(t,s)$ is a family of continuous, closed curves, where the curves are paremeterised by t and the family by s, such that the first curve in the family (when $s = 0$) is the original curve γ and the final 'curve' in the family (when $s = 1$) is a single point. More informally, F shows us specifically how to continuously deform γ to a single point within the region.

(It is probably worth pointing out here that the definition of simply connected as meaning 'without holes' only works in two dimensions. If we consider a ball, for example, and remove a single point, the resulting set clearly has a whole, but it is also clearly possible to shrink any continuous curve to a point regardless of the hole. (Think about it for a bit if it isn't clear!) These 'higher-dimensional' holes lead to so-called 'higher homotopy groups', or, more tractably, to homology theory – which I think actually originated in the study of functions of a complex variable!)

18. Complex integration. We now enter into one of the core parts of the course, the notion of *contour integrals* in the complex plane. We shall introduce these in the same way as done in Goursat (§25) and then show how they may be computed by reducing to the line integrals one studies in multivariable calculus.

Recall that in one-variable calculus we define the definite integral of a function f between points a and b more or less as follows:

$$\int_a^b f(x)\,dx = \lim_{\Delta x_k \to 0} \sum_{k=1}^n f(x_k^*)\Delta x_k,$$

where $\Delta x_k = x_k - x_{k-1}$ and $x_k^* \in [x_{k-1}, x_k]$.[3] Now when we defined derivatives of functions on the complex plane, we were able to proceed by using essentially the same definition we used in the case of real-variable calculus. Let us see whether the same thing can be done in this case. Thus, let a and b be two *complex* numbers, and consider how we might adapt the limit definition above to this case. First of all we need to determine what is meant by the intermediate points between a and b; evidently we need a set of values z_1, z_2, $\cdots$, z_{n-1}. Now in the real case there is no real point in doing anything except going directly from the initial point to the final point; but in the complex plane there are many different paths which lead from a to b, and from what we have seen so far it is possible that these different paths may lead somehow to different results. Thus we evidently need to pick a path. Suppose that $\gamma(t)$ is a smooth (continuous with continuous derivative) path from a to b, and let $z_0 = a$, z_1, z_2, $\cdots$, z_{n-1}, $z_n = b$ be points along the curve ordered

[3] More precisely, for those of you who know something of ϵ–δ definitions, the limit above can be defined as follows: we say it is equal to L if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every *partition* $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$, $x_0 = a < x_1 < \cdots < x_n = b$ satisfying $\max\{|\Delta x_k|\,|k = 1, 2, \cdots, n\} < \delta$ and any set $\{x_k^* | k = 1, \cdots, n\}$ satisfying $x_k^* \in [x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^n f(x_k^*)\Delta x_k - L \right| < \epsilon.$$

(Here, of course, $\Delta x_k = x_k - x_{k-1}$.)

by increasing parameter value. (Cf. Figure 12 in §25 of Goursat.) Then we define $\Delta z_k = z_k - z_{k-1}$, and consider the sum

$$\sum_{k=1}^{n} f(z_k^*)\Delta z_k$$

(where z_k^* is some point along the curve between z_{k-1} and z_k in parameter value). Now suppose that

$$f(x+iy) = P(x,y) + iQ(x,y)$$

and write

$$z_k^* = x_k^* + iy_k^*, \qquad \Delta z_k = \Delta x_k + i\Delta y_k;$$

then working out the above product, we have

$$\begin{aligned}
\sum_{k=1}^{n} f(z_k^*)\Delta z_k &= \sum_{k=1}^{n}[P(x_k^*, y_k^*) + iQ(x_k^*, y_k^*)][\Delta x_k + i\Delta y_k] \\
&= \sum_{k=1}^{n} P(x_k^*, y_k^*)\Delta x_k - Q(x_k^*, y_k^*)\Delta y_k + i[P(x_k^*, y_k^*)\Delta y_k + Q(x_k^*, y_k^*)\Delta x_k] \\
&= \sum_{k=1}^{n} P(x_k^*, y_k^*)\Delta x_k - Q(x_k^*, y_k^*)\Delta y_k + i\sum_{k=1}^{n} P(x_k^*, y_k^*)\Delta y_k + Q(x_k^*, y_k^*)\Delta x_k.
\end{aligned}$$

Now we wish to consider the limit of the above sum as $\Delta z_k \to 0$ (in the same sense as elaborated in the footnote above). Since $\Delta z_k = \Delta x_k + i\Delta y_k$, this is the same as the limit as Δx_k and Δy_k go to zero independently. Thus we may write

$$\begin{aligned}
\lim_{\Delta z_k \to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k = \lim_{(\Delta x_k, \Delta y_k)\to(0,0)} &\sum_{k=1}^{n} P(x_k^*, y_k^*)\Delta x_k - Q(x_k^*, y_k^*)\Delta y_k \\
&+ i\sum_{k=1}^{n} P(x_k^*, y_k^*)\Delta y_k + Q(x_k^*, y_k^*)\Delta x_k
\end{aligned}$$

,

and we recognise these limits as giving ordinary line integrals of the form we have studied previously. In particular, the above limit becomes

$$\int_\gamma P(x,y)\,dx - \int_\gamma Q(x,y)\,dy + i\left[\int_\gamma P(x,y)\,dy + \int_\gamma Q(x,y)\,dx\right];$$

assuming, of course, that all of these integrals exist (as they will if P and Q are both continuous, for example). We take this as our definition and write

$$\int_\gamma f(z)\,dz = \int_\gamma P(x,y)\,dx - \int_\gamma Q(x,y)\,dy + i\left[\int_\gamma P(x,y)\,dy + \int_\gamma Q(x,y)\,dx\right].$$

Now suppose that γ is parameterised as $\gamma(t) = (x(t), y(t))$, $t \in [t_0, t_1]$; then the above can be written

$$\int_{t_0}^{t_1} P(x(t),y(t))x'(t) - Q(x(t),y(t))y'(t) + i[P(x(t),y(t))y'(t) + Q(x(t),y(t))x'(t)]\,dt.$$

Note that this is exactly what we would obtain if we were to replace dz in the integral $\int_\gamma f(z)\,dz$ with $x'(t)dt + iy'(t)dt$ and integrate from t_0 to t_1, i.e., if we were to pretend that the complex integral were simply another line integral with element $x'(t)dt + iy'(t)dt$. While this is not in itself a proof of anything, of course, it is useful for remembering the above formula; and it also suggests another mode of calculation: suppose that the curve γ is written in *complex* form as $z(t) = x(t) + iy(t)$; then the integral $\int_\gamma f(z)\,dz$ is equal to

$$\int_{t_0}^{t_1} f(x(t) + iy(t))(x'(t) + iy'(t))\,dt = \int_{t_0}^{t_1} f(z(t))z'(t)\,dt,$$

where we are useful to use *complex* techniques to determine $z'(t)$ and $f(z(t))$. In other words, this formula does not require us to split f into its real and imaginary parts, which is not convenient in many cases (such as when f is most usefully represented in terms of polar coordinates, for example).

The integral $\int_\gamma f(z)\,dz$ is called a *contour integral*.

19. First glimpse of the Cauchy integral theorem. Let us consider what happens when we integrate an analytic function over a closed curve. More specifically, suppose that we have a function $f(x + iy) = P(x, y) + iQ(x, y)$ which is analytic over a simply connected region U which has boundary curve C, and assume that C is oriented with respect to U as required by Green's theorem. Let us assume furthermore that the real and imaginary parts of f, namely P and Q, have continuous first-order partial derivatives. Then by applying Green's theorem and the Cauchy-Riemann equations, we have

$$\begin{aligned}\int_C f(z)\,dz &= \int_C P(x, y)\,dx - Q(x, y)\,dy + i\int_C Q(x, y)\,dx + P(x, y)\,dy \\ &= \int_U \frac{\partial}{\partial x}[-Q(x, y)] - \frac{\partial}{\partial y}[P(x, y)]\;dA + \int_U \frac{\partial}{\partial x}[P(x, y)] - \frac{\partial}{\partial y}[Q(x, y)]\;dA \\ &= \int_U -\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\,dA + \int_U \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\,dA = 0,\end{aligned}$$

since the Cauchy-Riemann equations give

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}.$$

Thus under the assumptions above, the contour integral of an analytic function over a closed curve is always 0. This is a central result in complex variable theory.

Unfortunately, the above demonstration, since it requires that the partial derivatives of the real and imaginary parts of f be continuous, is not sufficient for our purposes, since we actually want to use this result to *prove* continuity of those derivatives. Thus we shall soon see another proof of this result from first principles which does not make use of this assumption.