Summary:

- We continue to discuss conformal mappings and expand on a couple examples from Goursat.

16. Examples of conformal maps. (a) (See Goursat, §19, Example 2.) Consider the map on the punctured plane $\mathbf{R}^2\backslash\{(0,0)\}$ which is given in complex notation by

$$f(z) = \frac{1}{z}.$$

Since this function is analytic on the punctured plane, it must be conformal at every point other than the origin. Let us consider how it behaves with respect to the unit circle. We have the following properties:

$$\text{If } |z| = 1 \text{ then } |f(z)| = \left|\frac{1}{z}\right| = \frac{1}{|z|} = 1.$$
$$\text{If } |z| > 1 \text{ then } |f(z)| = \left|\frac{1}{z}\right| = \frac{1}{|z|} < 1.$$
$$\text{If } |z| < 1 \text{ then } |f(z)| = \left|\frac{1}{z}\right| = \frac{1}{|z|} > 1.$$

This means that the map f takes the unit circle to itself, while it takes the region *outside* the unit circle to the region *inside* the unit circle, and vice versa. See Fig. 1. It is worth noting that on the unit circle

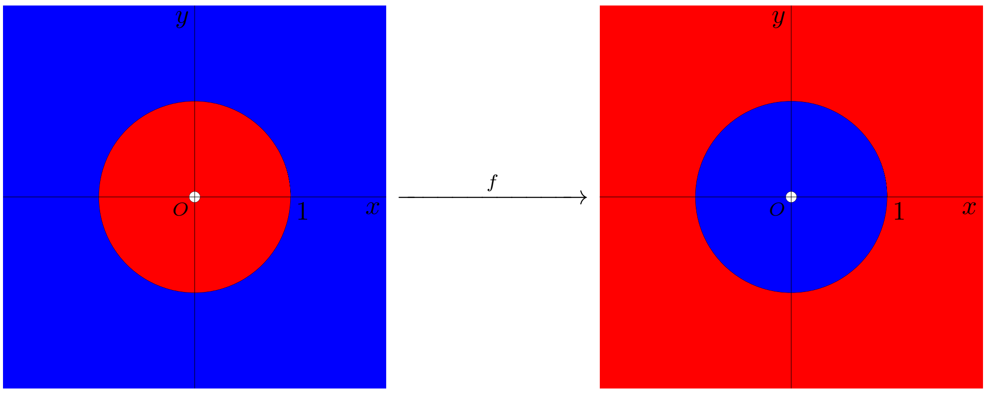

FIG. 1

$$f(z) = \frac{1}{z} = \overline{z}.$$

Note though that $\overline{z}$ is *not* an analytic function in general! It does turn out to be (almost) conformal though (it preserves magnitudes of angles but reverses their sense); and it can be shown (see §21 of Goursat, noting that replacing Q by $-Q$ is equivalent to taking the complex conjugate of f) that every sufficiently smooth conformal map is either an analytic function or the conjugate of an analytic function (which is the same thing as an analytic function of $\overline{z}$, as is apparent if one thinks of a Taylor expansion: but that is a bit beyond what we have technically covered so far).

(b) (See Goursat, §22, Example 2.) Let us now consider the function on the entire plane given in complex notation by

$$f(x+iy) = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y.$$

This function is analytic everywhere, and will be conformal everywhere that its derivative is nonzero. (We pause for a moment to clarify a point which the author fumbled during lecture. The derivative of $\cos z$ is $-\sin z$, which means that $\cos z$ will be conformal at every point where $\sin z$ is nonzero. Now

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y;$$

and for this to be zero, we see first of all that we must have $\sin x = 0$ (since $\cosh y \geq 1$ for all real y), and since this means that $\cos x \neq 0$, we must have $\sinh y = 0$, or $y = 0$. Thus the zeros of $\sin z$ over the complex plane are the same as those of $\sin x$ over the real line, i.e., $n\pi$, $n \in \mathbf{Z}$.) Thus f will be conformal at every point inside the strip $\{x+iy|0<x<\pi, y>0\}$. Let us consider how f maps straight lines within this strip. Let us consider first a horizontal line, say $y = y_0 > 0$. On such a line, f is equal to

$$f(x+iy_0) = \cos x \cosh y_0 - i \sin x \sinh y_0,$$

where $x \in (0,\pi)$. Now this is just another way of writing the parametric curve

$$t \mapsto (\cosh y_0 \cos t, -\sinh y_0 \sin t), \qquad t \in (0,\pi).$$

If we denote this curve by $(x(t), y(t))$ (where unfortunately here $x(t)$ and $y(t)$ are completely distinct from the real and imaginary parts of z), then we have

$$\left(\frac{x(t)}{\cosh y_0}\right)^2 + \left(\frac{y(t)}{\sinh y_0}\right)^2 = 1,$$

i.e., the curve must lie on an ellipse with major axis $\cosh y_0$ along the horizontal axis and minor axis $\sinh y_0$ along the vertical axis, and centred at the origin. Now since $y_0 > 0$, $\sinh y_0 > 0$, so $-\sinh y_0 < 0$ and $y(t) < 0$ for all $t \in (0,\pi)$, while $x(t)$ takes on all values from $\cosh y_0$ to $-\cosh y_0$. Thus we obtain the lower half of this ellipse.

Now let us consider a vertical line, say $x = x_0 \in (0,\pi)$. Working as before, we see that on this line

$$f(x_0+iy) = \cos x_0 \cosh y - i \sin x_0 \sinh y.$$

If $x_0 = \pi/2$ then $\cos x_0 = 0$ and this is simply a parametrisation of the negative imaginary axis. Otherwise, we again write

$$(x(t), y(t)) = (\cos x_0 \cosh t, -i \sin x_0 \sinh t), \qquad t \in (0,\pi)$$

and note that (this follows from the basic identity $\cosh^2 x - \sinh^2 x = 1$)

$$\left(\frac{x(t)}{\cos x_0}\right)^2 - \left(\frac{y(t)}{\sin x_0}\right)^2 = 1,$$

which means that the curve lies on a hyperbola opening along the real axis with intercept $\pm\cos x_0$ and with asymptotes having slope $\pm\tan x_0$. Now we note that $y(t) < 0$ for all t, while $x(t) > 0$ for $t \in (0,\pi/2)$ and $x(t) < 0$ for $t \in (\pi/2,\pi)$; thus in the first case we have the lower right-hand portion of the hyperbola, while in the second case we have the lower left-hand portion. See Fig. 2. Note especially how the blue and red

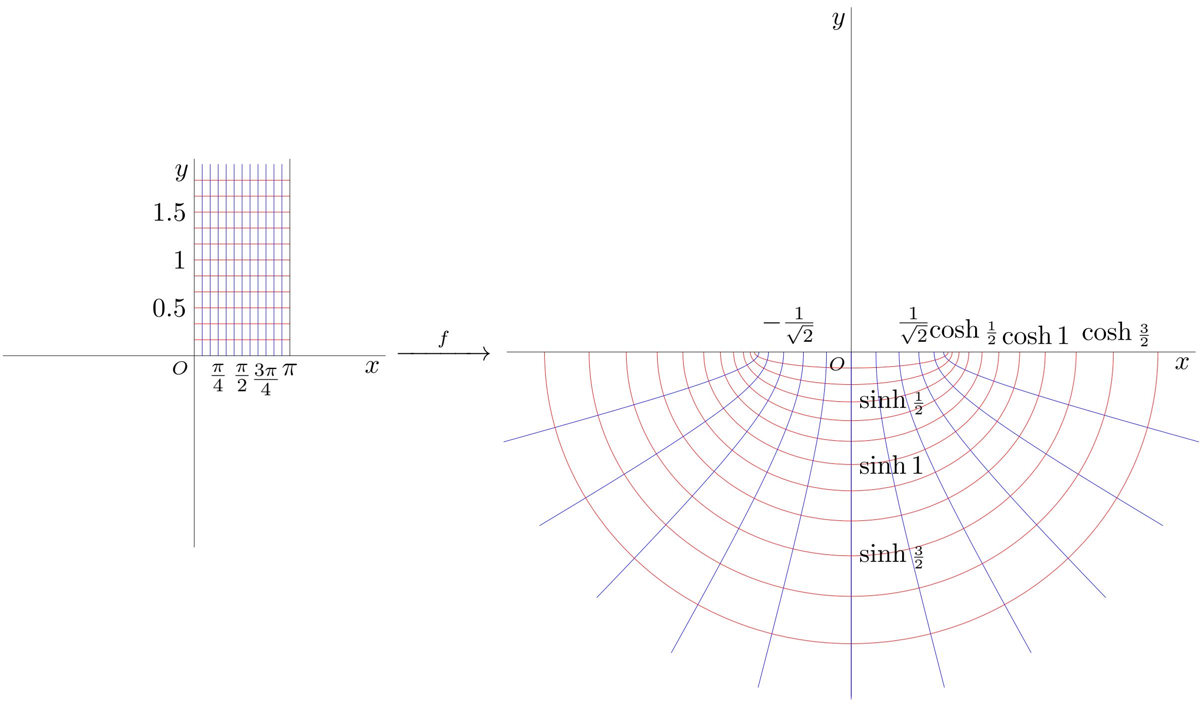

FIG. 2

curves on the right intersect at right angles, exactly like those on the left.