Summary:

- We discuss the branches of the logarithm function defined previously and show how to differentiate them.
- We introduce the extension of the trigonometric functions to the complex plane, and relate them to the ordinary trigonometric and hyperbolic trigonometric functions of a real variable.
- We show how the inverse trigonometric functions can be determined in terms of roots and logarithms, and calculate their derivatives.
- Finally, we give a slightly more careful description of the kind of region we assume our functions are defined; then we give an introduction to *conformal mappings* and show that analytic functions are conformal.

12. Differentiation of Log. Recall that we have defined the complex logarithm as a multi-valued function as follows. If z is any nonzero complex number and $re^{i\theta}$ is any polar representation of z, then we define

$$\operatorname{Log} z = \log r + i(\theta + 2n\pi), \qquad n \in \mathbf{Z},$$

where here log denotes the ordinary real logarithm of a positive real number. (Note that this definition allows us to extend the logarithm to negative real numbers but *not* to zero. Since even over the complex plane the exponential is never 0, there is no way to extend the logarithm to zero.) As for the root functions we studied previously, a single-valued, continuous logarithm can only be defined on a cut plane. Let us see how this works in practice. Suppose that we cut the plane along the ray $\theta = \theta_0$, i.e., that we define the logarithm only on complex numbers with polar representation $z = re^{i\theta}$ where $\theta \in (\theta_0, \theta_0 + 2\pi)$, and that we consider only this polar representation in defining the logarithm. (Note that, while related, these are two distinct points.) Then we have

$$\operatorname{Log} z = \log r + i\theta.$$

We note that this function is continuous on the cut plane; an outline of a proof is given in the appendix. Some examples related to this are given in the problem set.

Let us now see whether these branches of Log are analytic functions. Specifically, let us take the above branch, obtained by cutting the plane along $\theta = \theta_0$. We shall denote this particular branch by $\operatorname{Log} z$ in the following, for convenience. We must determine whether the limit

$$\lim_{h \to 0} \frac{\operatorname{Log}(z+h) - \operatorname{Log}(z)}{h}$$

exists. This limit may clearly be written as

$$\lim_{z' \to z} \frac{\operatorname{Log} z' - \operatorname{Log} z}{z' - z}.$$

Now if $z = re^{i\theta}$, where $\theta \in (\theta_0, \theta_0 + 2\pi)$, then as long as z' is close enough to z[1] we may write $z' = r'e^{i\theta'}$ where $\theta' \in (\theta_0, \theta_0 + 2\pi)$ and also θ' is close to θ. Let us now define

$$w = \operatorname{Log} z = \log r + i\theta, \qquad w' = \operatorname{Log} z' = \log r' + i\theta'.$$

Then

$$\frac{\operatorname{Log} z' - \operatorname{Log} z}{z' - z} = \frac{w' - w}{e^{w'} - e^{w}}.$$

Now as $z' \to z$, we have clearly (by continuity of the logarithm) $\operatorname{Log} z' \to \operatorname{Log} z$, i.e., $w' \to w$; and in this limit the above fraction becomes

$$\lim_{w' \to w} \frac{w' - w}{e^{w'} - e^{w}} = \lim_{w' \to w} \frac{1}{\frac{e^{w'} - e^{w}}{w' - w}} = \frac{1}{\lim_{w' \to w} \frac{e^{w'} - e^{w}}{w' - w}} = \frac{1}{e^{w}},$$

[1] Specifically, we need the angle between them to be less than the smaller of $\theta - \theta_0$ and $\theta_0 + 2\pi - \theta$.

since the exponential function is analytic and is equal to its own derivative. But recall that

$$e^w = e^{\operatorname{Log} z} = z,$$

so that we have shown that

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}.$$

Note that this final result does not depend on the choice of branch cut; in other words, each branch of Log has the same derivative. This accords with what we know about derivatives from ordinary calculus, since the various branches of Log differ only by constants.

To sum up, we have shown that each branch of Log is an analytic function on its domain, and all of the branches have the same derivative, namely $1/z$.

Appendix I. Continuity of Log. Let us show that each branch of the logarithm, as outlined at the start of the section above, is in fact continuous. We shall give a formal ϵ-δ argument, but provide intuitive commentary to hopefully make the ideas clear to those who do not have much background in such things. Thus let $z = re^{i\theta}$ be an element of the cut plane, with $\theta \in (\theta_0, \theta_0 + 2\pi)$, and let $\epsilon > 0$. We may assume that $\epsilon < \frac{\pi}{4}$. Since log is continuous on the positive real line, there must be a $\delta' > 0$ such that

$$|\log r - \log r'| < \frac{1}{2}\epsilon \qquad \text{if} \qquad |r - r'| < \delta';$$

in other words, if r' is close to r then $\log r'$ is close to $\log r$. Further, it can be shown that the function $z \mapsto |z|$ is continuous; thus there is a $\delta'' > 0$ such that

$$||z| - |z'|| < \delta' \qquad \text{if} \qquad |z - z'| < \delta'';$$

in other words, $|z|$ is close to $|z'|$ if z is close to z' (clearly a reasonable statement geometrically!). Dealing with the angular part of z and z' is slightly messy; intuitively though the result is clear: if z' is sufficiently close to z, then we may write $z' = r'e^{i\theta'}$ where $\theta' \in (\theta_0, \theta_0 + 2\pi)$ and θ' is close to θ. To prove what we need carefully, though, let us set

$$\delta''' = \begin{cases} \frac{1}{2} r \sin(\theta - \theta_0), & \theta \in (\theta_0, \theta_0 + \pi/2) \cup (\theta_0 + 3\pi/2, \theta_0 + 2\pi), \\ \frac{1}{2} r, & \text{otherwise.} \end{cases}$$

Since $2\delta'''$ is simply the distance from z to the cut (draw a picture!), it is clear that $|z - z'| < \delta'''$ means that z' is on the same side of the cut as z, and hence can be written in the above form. Now let δ be the smaller of δ', δ'', δ''', and $\sin(\epsilon/2)$, and suppose that

$$|z - z'| < \delta.$$

By the foregoing, then,

$$||z| - |z'|| < \delta', \qquad \text{so} \qquad |\log |z| - \log |z'|| < \frac{1}{2}\epsilon;$$

furthermore, writing $z' = r'e^{i\theta'}$, $\theta' \in (\theta_0, \theta_0 + 2\pi)$, it is clear geometrically (again, draw a picture!) that the angle between z and z' is no greater than $\arcsin \delta$, which is bounded by $\epsilon/2$, so that $|\theta - \theta'| < \epsilon/2$. Thus finally

$$|\operatorname{Log} z - \operatorname{Log} z'| = |\log r + i\theta - \log r' + i\theta'| \le |\log |z| - \log |z'|| + |\theta - \theta'| < \epsilon,$$

proving continuity of Log, as desired.

13. Trigonometric functions. To extend the trigonometric functions to the complex plane, we shall proceed in the same way we did with the exponential function. Recall that on the real line we have the power series expansions

$$\sin x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k x^{2k+1}, \qquad \cos x = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k x^{2k}.$$

Since the radius of convergence of both of these series is infinite, they must converge on the entire complex plane as well; thus we may define

$$\sin z = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k z^{2k+1}, \qquad \cos z = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k z^{2k},$$

where now z is any complex number. Moreover, as we mentioned in our discussion of the exponential function in section 11 above, these power series are the *unique* way of extending sin and cos to the complex plane as analytic functions.

The standard identities of trigonometry can be shown to hold over the complex numbers as well; in particular, we have

$$\cos^2 a + \sin^2 a = 1,$$
$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b, \qquad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,$$
$$\sin 2a = 2 \sin a \cos a, \qquad \cos 2a = \cos^2 a - \sin^2 a,$$

and so forth, where now a and b can be any complex numbers. Moreover, sin is odd ($\sin(-z) = -\sin z$) while cos is even ($\cos(-z) = \cos z$), as with real numbers. Further, the differentiation formulæ for sin and cos also hold. This can be shown by differentiating the above series:[2]

$$\frac{d}{dz}\sin z = \frac{d}{dz}\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k z^{2k+1} = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k z^{2k} = \cos z,$$
$$\frac{d}{dz}\cos z = \frac{d}{dz}\sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k z^{2k} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)!}(-1)^k z^{2k-1} = -\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k z^{2k+1} = -\sin z,$$

where we have set the lower index to 1 in the second series on the second line since the constant term in the series for $\cos z$ differentiates to zero, and we have adjusted the index in the last equality.

Now recall that, by substituting in to the power series expression for e^z, we found that when y is real

$$e^{iy} = \cos y + i \sin y.$$

Now there is nothing in this derivation which requires y to be a real number; thus with the above definitions for sin and cos, we find that for all *complex* numbers z that

$$e^{iz} = \cos z + i \sin z.$$

Using the fact that cos is odd and sin is even, we see that

$$e^{-iz} = \cos(-z) + i \sin(-z) = \cos z - i \sin z;$$

adding and subtracting these two equations, we obtain the results

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

This allows us to derive expressions for the real and imaginary parts of $\cos z$ and $\sin z$. First of all, note that if y is real (actually for all complex y if we define cosh and sinh in the usual way, but we are only interested in real y for the moment)

$$\cos iy = \frac{e^{-y} + e^{y}}{2} = \cosh y, \qquad \sin iy = \frac{e^{-y} - e^{y}}{2i} = i \sinh y,$$

[2] As noted previously, convergent power series can be differentiated term-by-term on their discs of convergence.

where as usual

$$\cosh y = \frac{e^y + e^{-y}}{2}, \qquad \sinh y = \frac{e^y + e^{-y}}{2}.$$

Thus if $z = x + iy$,

$$\begin{aligned}\cos z &= \cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y,\\ \sin z &= \sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y.\end{aligned}$$

Now since cosh and sinh are unbounded, this means in particular that cos and sin are unbounded along the imaginary direction. In particular, the inequalities $|\cos x| \leq 1$, $|\sin x| \leq 1$, which are true for real x, do *not* hold for complex numbers.

Similar results can be derived for the other trigonometric functions (tangent, cotangent, secant, and cosecant) but we shall not go into that here.

14. Inverse trigonometric functions. Let us see what we can find about the inverse trigonometric functions, given the foregoing. Let us first consider $\sin z$; or, since we are interested in finding its inverse, $\sin w$, where w is another complex variable. We have the relation

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

Now let us set $z = \sin w$ and see whether we can solve for w. We have

$$\begin{aligned}\frac{e^{iw} - e^{-iw}}{2i} &= z\\ e^{iw} - e^{-iw} &= 2iz\\ e^{2iw} - 1 &= 2ize^{iw}\\ e^{2iw} - 2ize^{iw} - 1 &= 0\\ e^{iw} &= \frac{1}{2}\left(2iz + \left(4(iz)^2 + 4\right)^{1/2}\right)\\ &= iz + \left(1 - z^2\right)^{1/2},\end{aligned}$$

where we have dispensed with the $\pm$ usually present in the quadratic formula since $\left(1 - z^2\right)^{1/2}$ is defined to mean both square roots. Thus we may write

$$w = \frac{1}{i}\mathrm{Log}\left[iz + \left(1 - z^2\right)^{1/2}\right].$$

In other words, whenever w is any of the (infinitely many) complex numbers indicated by the right-hand side of this equation, we must have $\sin w = z$. We thus define

$$\arcsin z = \frac{1}{i}\mathrm{Log}\left[iz + \left(1 - z^2\right)^{1/2}\right].$$

Note that there are, in general, *two* distinct sources of multi-valuedness in the above expression, one from the square root (when $z \neq \pm 1$) and the other from the log. This is in good accord with our understanding of the graph of $\sin x$ on the real line: as long as $y_0 \neq \pm 1$, the graph of $y = \sin x$ will intersect the line $y = y_0$ twice per interval of length 2π.

Similar expressions can be derived for arccos and arctan but we pass over them for the moment.

The above expression may be differentiated, assuming that we are using appropriate branches:

$$\begin{aligned}\frac{d}{dz}\frac{1}{i}\mathrm{Log}\left[iz + \left(1 - z^2\right)^{1/2}\right] &= \frac{1}{i}\frac{1}{iz + (1-z^2)^{1/2}}\left(i - \frac{z}{(1-z^2)^{1/2}}\right)\\ &= \frac{1}{iz + (1-z^2)^{1/2}}\frac{\left(1-z^2\right)^{1/2} + iz}{(1-z^2)^{1/2}} = \frac{1}{(1-z^2)^{1/2}},\end{aligned}$$

in accord with what we know from real-variable calculus (except recall that here the square root means *both* square roots, i.e., it has a sign ambiguity).

15. Regions; conformal mappings. We have mentioned that we are principally interested in functions which are analytic in some *region*, rather than at a single point. We have however not defined what kind of region we are interested in. We are interested in the first place in functions which are analytic everywhere inside a so-called *simple closed curve*, i.e., a closed curve which does not intersect itself; such a region is *simply-connected* in the sense in which that word is typically used in discussions of Green's theorem, namely, it does not have any *holes*.[3] Later we shall also consider functions which are analytic on a set which has a finite number of holes, i.e., whose boundary is a finite number of simple closed curves, which moreover do not intersect each other. Whenever we speak of an analytic function, we are assuming that the function is analytic throughout a region of this form.

We shall now introduce so-called *conformal mappings*. It will turn out that all analytic functions on the complex plane are conformal mappings whenever they have nonzero derivative, but the definition of a conformal mapping does not require any use of complex numbers. A map

$$f : \mathbf{R}^2 \to \mathbf{R}^2$$

is said to be *conformal* at a point p when it preserves angles at that point; in other words, it $\gamma_1(t)$ and $\gamma_2(t)$ are any two curves which intersect at p, which for convenience and without loss of generality we may take to be $t = 0$ for both curves, then the angle between $\gamma_1(t)$ and $\gamma_2(t)$ at $t = 0$ is equal to the angle between $f(\gamma_1(t))$ and $f(\gamma_2(t))$ at $t = 0$, in both magnitude and sign (i.e., we measure it in the same direction, either clockwise or counterclockwise).[4] (See figures $9a$ and $9b$ in Goursat for an illustration.) Note that, in general, a map must be at least differentiable (in the sense of real functions on the plane!) for the angle of the image curves to make sense. Some examples immediately come to mind.

EXAMPLES. 1. Since translations and rotations of the plane preserve distances, they also preserve angles, and hence give conformal transformations.

2. So-called *isotropic scalings* of the plane, i.e., maps

$$(x, y) \mapsto (ax, ay),$$

where $a = 0$, are also conformal maps. This will follow from our general result below.

The main application we shall make of conformal mappings is to find solutions of Laplace's equation, which we shall take up probably in the second half of the course. The main example of conformal maps for us is given by the following result:

If f is analytic and $f'(z_0) \neq 0$, then f is conformal at z_0.

This may be shown as follows. (Here we first give the derivation given in the lecture, and supplement it to fill in a hole; we follow this with a slightly more concise demonstration.) For convenience we treat complex numbers as though they were their corresponding points in the plane. Let $\gamma_1(t)$ and $\gamma_2(t)$ be two smooth curves which satisfy $\gamma_1(0) = \gamma_2(0) = z_0$. Then they have tangent vectors there

$$\mathbf{T}_1 = \gamma_1'(0), \qquad \mathbf{T}_2 = \gamma_2'(0),$$

and hence make an angle θ which satisfies

$$\cos\theta = \frac{\mathbf{T}_1 \bullet \mathbf{T}_2}{|\mathbf{T}_1||\mathbf{T}_2|},$$

[3] For those who have seen something of general topology, the main point is that we are interested in functions which are analytic on some connected, simply-connected open set.

[4] For those of you who know something of modern differential geometry, the curves $\gamma_1(t)$ and $\gamma_2(t)$ here are being used as proxies for tangent vectors.

where $\bullet$ denotes the dot product. Now since f is analytic, it is in particular differentiable (in the real-variable sense) as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$, and thus the curves $f \circ \gamma_1$ and $f \circ \gamma_2$ are also smooth; moreover they have tangent vectors

$$\mathbf{S}_1 = f'(z_0) \cdot \gamma_1'(0), \qquad \mathbf{S}_2 = f'(z_0) \cdot \gamma_2'(0),$$

where we treat γ_1 and γ_2 as though they were complex-valued, and $\cdot$ denotes multiplication of complex numbers. (The foregoing is a simple extension of the chain rule.) Thus the angle between these image curves, say θ', satisfies

$$\cos\theta' = \frac{\mathbf{S}_1 \bullet \mathbf{S}_2}{|\mathbf{S}_1||\mathbf{S}_2|}.$$

Now recall (see the first example in §2, notes of May 5, above) that if z and w are any two complex numbers, then the dot product of the vectors corresponding to z and w is equal to $\operatorname{Re} \overline{z} w$. Thus we may compute as follows:

$$\begin{aligned}\mathbf{S}_1 \bullet \mathbf{S}_2 &= \operatorname{Re} \overline{f'(z_0)\mathbf{T}_1} f'(z_0)\mathbf{T}_2 = \operatorname{Re} \overline{f'(z_0)} f'(z_0) \overline{\mathbf{T}_1}\mathbf{T}_2 \\ &= |f'(z_0)|^2 \operatorname{Re} \overline{\mathbf{T}_1}\mathbf{T}_2 = |f'(z_0)|^2 \, \mathbf{T}_1 \cdot \mathbf{T}_2.\end{aligned}$$

Since $|\mathbf{S}_1|$ can be computed in terms of a dot product, we see that

$$\begin{aligned}\cos\theta' &= \frac{\mathbf{S}_1 \bullet \mathbf{S}_2}{|\mathbf{S}_1||\mathbf{S}_2|} = \frac{|f'(z_0)|^2 \, \mathbf{T}_1 \cdot \mathbf{T}_2}{|f'(z_0)| \, |\mathbf{T}_1| \, |f'(z_0)| \, |\mathbf{T}_2|} \\ &= \frac{\mathbf{T}_1 \cdot \mathbf{T}_2}{|\mathbf{T}_1||\mathbf{T}_2|} = \cos\theta.\end{aligned}$$

This shows that θ and θ' have the same cosine. However this of course does not mean that they are equal. (This point was not mentioned in the lecture.) To show that they are actually equal, we recall also that if z and w are any two complex numbers, the *cross product* (more carefully, the $\mathbf{k}$ component of the cross product) of z and w is equal to $\operatorname{Im} \overline{z} w$. Now recall from vector calculus that the cross product in this case is also given by $|z||w| \sin\phi$, where ϕ is the angle between the vectors corresponding to z and w. The foregoing calculation shows, replacing Re by Im everywhere, that we must have $\sin\theta = \sin\theta'$. Since two angles which have the same sine and cosine must be equal up to some integer multiple of 2π, and this means for our purposes that they are the same angle, this shows that f must be conformal at z_0, as claimed.

A slightly more concise demonstration may be given as follows. (Those of you who are familiar with derivatives considered as linear maps can skip straight to the appendix where an even more concise proof is given.) Let $t > 0$ be small. Then the tangent vectors to γ_1 and γ_2 at $t = 0$, i.e., at z_0, can be approximated by

$$\frac{\gamma_1(t) - z_0}{t}, \qquad \frac{\gamma_2(t) - z_0}{t}.$$

Similarly, the tangent vectors to $f(\gamma_1(t))$ and $f(\gamma_2(t))$ can be approximated by

$$\frac{f(\gamma_1(t)) - f(z_0)}{t}, \qquad \frac{f(\gamma_2(t)) - f(z_0)}{t}.$$

Now for z near z_0 we may write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0),$$

where $o(z - z_0)$ denotes a quantity which vanishes faster than $z - z_0$ as the latter goes to zero; i.e.,

$$\lim_{z \to z_0} \frac{o(z - z_0)}{z - z_0} = 0.$$

Thus we have

$$f(\gamma_k(t)) = f(z_0) + f'(z_0)(\gamma_k(t) - z_0) + o(\gamma_k(t) - z_0),$$

so

$$\frac{f(\gamma_k(t)) - f(z_0)}{t} = f'(z_0)\frac{\gamma_k(t) - z_0}{t} + \frac{o(\gamma_k(t) - z_0)}{t}.$$

Now in the limit $t \to 0$ we have similarly $\gamma_k(t) = \gamma_k(0) + \gamma_k'(0)t + o(t) = z_0 + \gamma_k'(0)t + o(t)$, so that in this limit the last quantity on the right-hand side above vanishes and we find that the tangent vector to the curves $\gamma_k(t)$ are given by

$$f'(z_0)\gamma_k'(0),$$

where as before the multiplication is to be considered as multiplication of complex numbers. Now suppose that we have

$$\gamma_k'(0) = r_k e^{i\theta_k},$$

and that

$$f'(z_0) = re^{i\theta};$$

then the tangent vectors to the image curves are given by

$$f'(z_0)\gamma_k'(0) = rr_k e^{i(\theta_k+\theta)};$$

in other words, the effect of an analytic map f on tangent vectors to smooth curves is to scale and rotate, which clearly preserves angles. This shows that f is conformal at z_0, as claimed.

Appendix I. Abstract derivation. Let us consider f as a map of the real plane. Then its derivative $f'(z_0)$ is a linear map from the plane to itself which satisfies

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|),$$

where here $f'(z_0)$ is considered as a linear map and $z - z_0$ as a vector, and the 'product' above is the application of this linear map to this vector. Evidently, $f'(z_0)$ may be considered to be multiplication by the complex derivative also denoted $f'(z_0)$. Now abstractly the derivative as a linear map takes tangent vectors to tangent vectors; in other words, two tangent vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ (say) at the point z_0 are taken by the map f to the vectors $f'(z_0)\mathbf{T}_1$ and $f'(z_0)\mathbf{T}_2$. By the discussion in the last few lines of the section above, the angle between these vectors must be that between $\mathbf{T}_1$ and $\mathbf{T}_2$.

(I admit that this is a little bit hand-wavy. The reason for this is that the definition of 'conformal' given above is somewhat informal. The argument just given can be made entirely rigorous if we define 'preserves angles at a point' to mean that its derivative preserves angles as a map of tangent vectors, which is more or less equivalent to the definition in terms of curves given above.)