Summary:
- We clarify some matters related to branch cuts.
- We then fill in some points from the last set of lecture notes.
- Finally, we introduce power series and discuss how to extend the exponential and logarithm to complex numbers.

8. Roots and branch cuts, II. In the lecture notes from Tuesday, §7, we demonstrated that it is impossible to make a continuous choice of root on the entire complex plane, so that we either need to remove a part of the plane (make a *branch cut*) or embed the complex plane into a much larger set (the so-called *Riemann surface* of the function) in order to get a well-defined, continuous, single-valued function. In this section we will step back a bit to consider what all of this means, and why we are discussing it.

First of all, a philosophical point which will be useful to keep in mind at many other points in the course also. In mathematics there are some results or concepts which we study because they can be immediately used to solve problems, and there are other results or concepts which we study because they help deepen our understanding, even if they are not directly (or at least immediately) applicable to solving problems. In elementary calculus, for example, the product rule is of the first kind, as is the first derivative test; while the notion of a continuous function, or the extreme value theorem, are more of the second kind. In this class, methods for calculating residues, which we shall study later, are of the first kind; while branch cuts, which we are studying now, are of the second kind. We study them not so much because we need them immediately for applications, or because we can immediately solve problems about them, but because they help deepen our understanding of what an analytic function of a complex variable *is*, and how it might behave.[1]

With this in mind, let us go back and investigate exactly *why* we needed a branch cut in the first place. The most immediate answer is that we needed a branch cut to make sure we could keep our function continuous and single-valued. Why did it become multiple-valued in the first place?

Let z be any nonzero complex number, and suppose that $z = r(\cos\theta + i\sin\theta)$ is a polar form of z. Then clearly so is $r\left[\cos(\theta + 2\pi n) + \sin(\theta + 2\pi n)\right]$. Now consider the following diagram; the block on the left is to be read top to bottom, then left to right, and we use the abbreviation $\operatorname{cis}\theta$ for $\cos\theta + i\sin\theta$ (we will see very soon that $\operatorname{cis}\theta = e^{i\theta}$, of course):

$$\underbrace{\left.\begin{array}{lcccl}
\cdots, & r\operatorname{cis}(\theta - 2\pi m), & r\operatorname{cis}\theta, & r\operatorname{cis}(\theta + 2\pi m), & \cdots \\
\cdots, & r\operatorname{cis}(\theta - 2\pi(m-1)), & r\operatorname{cis}(\theta + 2\pi), & r\operatorname{cis}(\theta + 2\pi(m+1)), & \cdots \\
\cdots, & r\operatorname{cis}(\theta - 2\pi(m-2)), & r\operatorname{cis}(\theta + 4\pi), & r\operatorname{cis}(\theta + 2\pi(m+2)), & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots, & r\operatorname{cis}(\theta - 2\pi), & r\operatorname{cis}(\theta + 2\pi(m-1)), & r\operatorname{cis}(\theta + 2\pi(2m-1)), & \cdots
\end{array}\right\}}_{z} \overset{z\mapsto z^{1/m}}{\rightarrow} \left\{\begin{array}{c}
r^{\frac{1}{m}}\operatorname{cis}\frac{\theta}{m} \\
r^{\frac{1}{m}}\operatorname{cis}\frac{\theta+2\pi}{m} \\
r^{\frac{1}{m}}\operatorname{cis}\frac{\theta+4\pi}{m} \\
\vdots \\
r^{\frac{1}{m}}\operatorname{cis}\frac{\theta+2\pi(m-1)}{m}
\end{array}\right.$$

where each quantity on the left is equal to z, and where each line on the left maps under the mth root function to a single value on the right. The issue is that while each of the quantities on the left is a polar representation of the *same* complex number z, the m quantities on the right represent *distinct* complex numbers – namely, the m possible mth roots of z. This diagram indicates one way of looking at the issue: the mth root function is most naturally considered as acting on the polar representation of a complex number z, but it takes representations of the *same* complex number to representations of *distinct* complex numbers. The point of a branch cut is to allow us to single out a preferred choice of polar representation for z in such a way that the resulting mth root is uniquely defined. (In terms of the above diagram, such a choice corresponds to picking a specific row.)

For example, suppose that we take our branch cut along the positive real axis: then we may require the angle in any polar representation of z to lie in the interval $(0, 2\pi)$. Now suppose that we are given the complex number $z = -1$. Since the point corresponding to this number makes an angle of π radians with the positive real axis, we can write it as $z = \operatorname{cis}\pi$. Now we could equally well write $z = \operatorname{cis}(2k+1)\pi$ for

[1] I read recently somewhere – unfortunately I have forgotten where – that functions of a complex variable are essentially defined by their singularities. Of the three kinds of singularities we shall see in this course, namely poles, essential singularities, and branch points, branch points are the hardest to deal with; in other words, as far as singularities are concerned anyway, things get simpler from here on out!

any integer k; but our choice of interval $(0, 2\pi)$ for the angle requires us to use $z = \operatorname{cis}\pi$. The mth root we get in this case is then $z^{1/m} = \operatorname{cis}\pi/m$.

It is not hard to find other examples; we give two just to demonstrate the point. Suppose that we choose the same branch cut but now require the angle to lie in the interval $(2\pi, 4\pi)$; there is no reason why we can't do this. Then the point $z = -1$ will be represented as $z = \operatorname{cis} 3\pi$, and the corresponding choice of mth root will be $z^{1/m} = \operatorname{cis} 3\pi/m$.

Finally, suppose that we choose a different ray as our branch cut, say the positive *imaginary* axis. Our possible choices of intervals are different now: instead of avoiding the positive real axis, which has angle 0, we now need to avoid the positive imaginary axis, which has angle $\pi/2$. Thus we may choose an interval of the form $(-3\pi/2, \pi/2)$ (for example). In this case, the polar representation of z will be $z = \operatorname{cis}(-\pi)$, and the corresponding choice of mth root will be $z^{1/m} = \operatorname{cis}(-\pi/m)$.

To sum up: a branch cut determines the possible different choices of representation for z, and a selection of one of these makes the root function (or whatever other function we happen to be studying) single-valued.

Before moving on, I would like to emphasise again that the point of learning about branch cuts at this point is not because we are going to use them right away to solve problems (though we will see that they do come up in practical problems later on in the course), nor is it because we are going to immediately be able to go off and determine where functions have branch points. (Another, more involved, example of branch cuts is however given in the second part of §6 of Goursat.) Rather it is to be given an introduction to a particular feature of certain functions of a complex variable which we shall study more later.

– See §§5 and 6 above –

9. Conjugate harmonic functions [continuing §5]. Recall that we have shown in §5 above that the Cauchy-Riemann equations imply that the real and imaginary parts of an analytic function f must satisfy Laplace's equation[2] $\Delta u = 0$. However, the Cauchy-Riemann equations have more information than just this, as they give also a relationship between the real and imaginary parts. Thus we may consider the following question: Suppose that $P(x, y)$ is a real-valued function of two real variables which satisfies Laplace's equation; is there a function $Q(x, y)$ of two real variables such that

$$f(x + iy) = P(x, y) + iQ(x, y)$$

is an analytic function? It is not too hard to see that the answer is actually yes, at least if we stick to simply-connected regions. Let us write out the Cauchy-Riemann equations and see if we can solve them for Q:

$$\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}, \qquad \frac{\partial Q}{\partial y} = \frac{\partial P}{\partial x}. \tag{1}$$

Probably the most direct way to treat these equations is to use a bit of vector calculus. Let us define a vector field

$$\mathbf{F} = -\frac{\partial P}{\partial y}\mathbf{i} + \frac{\partial P}{\partial x}\mathbf{j};$$

then since P is harmonic we have

$$\operatorname{curl}\mathbf{F} = \frac{\partial}{\partial x}\frac{\partial P}{\partial x} - \frac{\partial}{\partial y}\left(-\frac{\partial P}{\partial y}\right) = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0,$$

which means that, as long as we stick to simply-connected regions (recall that these are regions 'without holes'; generally these are introduced when one studies Green's theorem), there must be a function $f(x, y)$ such that $\mathbf{F}(x, y) = \nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$. In other words, there must be a function f such that

$$\frac{\partial f}{\partial x} = -\frac{\partial P}{\partial y}, \qquad \frac{\partial f}{\partial y} = \frac{\partial P}{\partial x}.$$

[2] At least, assuming that they have continuous second-order partial derivatives. We shall see shortly that if a function f is analytic throughout a region – as opposed to at a single point – then this condition is always satisfied. As far as I know, functions which are analytic at isolated points are of interest only as mathematical curiousities, and have no particular use in applications, so we shall not generally worry about them.

But these are exactly the equations we wanted Q to satisfy; in other words, what we know from vector calculus shows us that there must be a solution Q to the equations (1). It is unique up to an additive constant.

To be more specific, recall that we also know from vector calculus that the function f can be written as

$$f(x, y) = \int_{(x_0,y_0)}^{(x,y)} \mathbf{F} \cdot d\mathbf{x} + C,$$

where (x_0, y_0) is any point in the domain of P, the integral is a line integral along any path joining the two points (it will not depend on this path because $\operatorname{curl} \mathbf{F} = 0$ implies that $\mathbf{F}$ is conservative) and C is any constant. (In vector calculus, of course, we take C to be a real constant. Here C can be any complex constant.) This allows us to write

$$Q(x, y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P}{\partial y}\, dx + \frac{\partial P}{\partial x}\, dy + C,$$

and finally

$$f(x + iy) = P(x, y) + i \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial P}{\partial y}\, dx + \frac{\partial P}{\partial x}\, dy + C.$$

10. Power series. Let us recall a few facts about power series over the real numbers. A *power series* is an infinite series of the form

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k, \tag{2}$$

where $\{a_k\}$ is a sequence of coefficients, x_0 is some real number, and we consider x as a variable real number. The series will be *absolutely convergent* (meaning that the sum of the absolute values of its terms will be finite)[3] on some interval of the form $(x_0 - R, x_0 + R)$, called the *interval of convergence*, where $R > 0$ is called the *radius of convergence* and can be calculated from

$$\frac{1}{R} = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|,$$

when this limit exists.

Suppose now that we allow the numbers in the series in (2) to become complex. Now it turns out that, just as for real numbers, a series of complex numbers which is absolutely convergent is also convergent, so we may begin by asking where this series is absolutely convergent, which means that we must consider the series

$$\sum_{k=0}^{\infty} |a_k| \, |z - z_0|^k \,.$$

But this is just a power series of *real* numbers with coefficients $|a_k|$, and must therefore converge when $|z - z_0| < R$, where R is given as before. From this we can draw two conclusions:

1. Power series over the complex numbers converge in *discs*;
2. In the case that the coefficients a_k are all real, the radius of the disc of convergence is equal to the radius of the interval of convergence.

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[3] The notion of absolute convergence is very important in more theoretical parts of analysis. Since a series of positive terms converges if and only if it has an upper bound, and since in most spaces in which these concepts make sense – and in particular, for real and complex numbers – an absolutely convergent series is convergent, we are to reduce a question of convergence of a series – which is hard – to the question of finding an upper bound for a series, which is generally simpler. We shall probably not have much need to use these concepts and results directly, however.

Point 2 in particular makes the term *radius* of convergence much more sensible!

Just as with real power series, power series of complex numbers can be added, multiplied (though that becomes messy very quickly, as anyone who has attempted such a procedure can surely attest!), and differentiated term-by-term. This means, *inter alia*, that power series represent analytic functions where they converge. Also as with real power series, a power series converges inside its disc of convergence and diverges outside; on the boundary, as with real power series, it may converge or diverge, depending on the point and the situation.[4] Our main interest with power series right now is that they provide a convenient way to extend the elementary transcendental functions (the exponential, trigonometric, and logarithmic functions) to complex numbers, which we take up now.

11. Exponentials and logarithms of a complex variable. Recall that the exponential function e^x has the power series representation

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k,$$

and that this series converges for all real numbers x. By our discussion above, this shows that the power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} z^k,$$

where z is now a *complex* variable, must converge for all complex numbers z. It is clearly equal to e^x when $z = x$ is a real number. Now it can be shown (and we shall probably be able to show this in the second half of the course) that analytic functions are incredibly *rigid*: roughly, if they are equal on any set which is not somehow 'discrete', they must be equal everywhere. (We shall make this more precise later as it is not exactly true as it stands.)[5] This suggests that the above power series of complex numbers, which as we have seen defines a function which is analytic everywhere on the complex plane, is the *unique* function analytic everywhere on the complex plane which is equal to the ordinary exponential function on the real axis. We thus define, for any complex number z, the complex exponential

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

When convenient for typographical reasons we may write $\exp z$ instead of e^z. The standard properties of exponential functions can be shown to follow from this expansion; for example, if z_1 and z_2 are any complex numbers, we have

$$\begin{aligned}
e^{z_1} e^{z_2} &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} z_1^k \right) \left(\sum_{\ell=0}^{\infty} \frac{1}{\ell!} z_2^\ell \right) \\
&= \sum_{k,\ell=0}^{\infty} \frac{1}{k!\ell!} z_1^k z_2^\ell \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} z_1^k z_2^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (z_1 + z_2)^n = e^{z_1+z_2},
\end{aligned}$$

[4] We note that it is possible to find a function which is analytic everywhere inside a disc but at no point of the boundary.

[5] For those who know enough topology to understand the following, we note that two analytic functions which agree on a set with at least one accumulation point must agree on the connected component of the intersection of their domains containing that set.

where in the third line we have introduced the variable $n = k + \ell$.

We know that on the real axis e^z agrees with the ordinary exponential function; what happens on the imaginary axis? Let $z = iy$; then we have

$$\begin{aligned} e^z = e^{iy} &= \sum_{k=0}^{\infty} \frac{1}{k!}(iy)^k \\ &= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!}(iy)^{2\ell} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}(iy)^{2m+1} \\ &= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!}(-1)^\ell y^{2\ell} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} i(-1)^m y^{2m+1} \\ &= \cos y + i \sin y, \end{aligned}$$

probably one of the most fascinating results in mathematics. This formula makes much of our work with powers and roots far more transparent: for example, the result

$$[r(\cos\theta + i\sin\theta)]^{1/m} = r^{1/m}(\cos\theta/m + i\sin\theta/m)$$

(where we have chosen just one particular mth root for simplicity) becomes now

$$\left[re^{i\theta}\right]^{1/m} = r^{1/m}e^{i\theta/m},$$

which is exactly what we would expect were the standard rules of exponents applicable to the complex exponential function.

Having now defined the exponential function for all complex numbers, we proceed to consider the logarithm. From what we have just seen, an arbitrary nonzero complex number z can be written in the form

$$z = re^{i\theta}$$

for some real number $r > 0$ ($r \neq 0$ since z is nonzero) and some real number θ. But since $r > 0$ we have

$$r = e^{\log r},$$

where here log represents the ordinary logarithm of positive real numbers; thus we can write

$$z = e^{\log r + i\theta}.$$

Now the defining property of the logarithm on real numbers is, that it is the inverse of the exponential function; if we wish to define the logarithm of a complex number the same way, the above formula suggests that we should define it to be $\log r + i\theta$. But here we run into the same problem we found when we discussed roots: θ is only defined up to an integer multiple of 2π. Thus for complex numbers we must evidently define the logarithm to be a multi-valued function. With this in mind, we define the logarithm of a nonzero complex number z, which we write $\operatorname{Log} z$, to be the collection of numbers

$$\operatorname{Log} z = \log r + i\theta,$$

where $r = |z|$ is the modulus of z and θ is any value of the argument of z. As with roots, this means that the logarithm has a branch point at the origin, and we must make a branch cut in order to get a single-valued continuous logarithm.

With these functions now defined, we may define exponents of any (nonzero) complex base and any complex power. First we recall that if $x_1 > 0$ and x_2 are two real numbers, we may write, by rules of exponents and logarithms (here log denotes the ordinary logarithm of positive real numbers)

$$e^{x_2 \log x_1} = e^{\log x_1^{x_2}} = x_1^{x_2}.$$

Now if we use the complex logarithm Log defined above, we can compute the left-hand side of the above equation for all *complex* numbers z_1 and z_2, as long as $z_1 \neq 0$. Thus, let $z_1 \neq 0$ and z_2 be two complex numbers; then we define

$$z_1^{z_2} = e^{z_2 \mathrm{Log}\, z_1}.$$

Note though that, since Log is multivalued, this definition in general makes $z_1^{z_2}$ a multivalued function as well. This leads to some rather amusing results. Let us give some examples.

EXAMPLES. 1. Before giving the amusing examples, let us first see how this definition fits in with the exponents we have already studied, namely integer powers and roots. If m is a positive integer and $z = r(\cos\theta + i\sin\theta)$ is any nonzero complex number, the above definition gives

$$z^m = e^{m \mathrm{Log}\, z} = \exp\left(m[\log r + i\theta]\right) = \exp\left(m\log r + im\theta\right) = e^{m\log r}e^{im\theta} = r^m(\cos m\theta + i\sin m\theta),$$

exactly in accord with our previous definition. Note that in this particular case the exponential function is *single*-valued, since if θ' is any other value of the argument of z, we would have $\theta' - \theta = 2\pi k$ for some integer k, and the above formula would give

$$r^m(\cos m\theta' + i\sin m\theta') = r^m(\cos m(2\pi k + \theta) + i\sin m(2\pi k + \theta)) = r^m(\cos m\theta + i\sin m\theta)$$

as before.

Let us now consider roots. Thus, again, let m be a positive integer and $z = r(\cos\theta + i\sin\theta)$ a nonzero complex number; then we have

$$z^{\frac{1}{m}} = \exp\left(\frac{1}{m}[\log r + i\theta]\right) = \exp\left(\frac{1}{m}\log r\right)\exp\left(i\frac{\theta}{m}\right) = r^{1/m}\left(\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\right),$$

exactly in accord with our original definition of mth roots. Recall that here θ represents *any* possible argument value for z, so that this expression represents all possible mth roots and is, as usual, multivalued for $m \neq 1$.

2. Now for some amusing examples. Let us recall that the exponential for real numbers is only defined for *positive* bases. We now have a means of defining it for arbitrary *complex* bases, but in particular for *negative* real bases; what does it give us? In particular, what is say -1 raised to an irrational power, say $\sqrt{2}$? To find this, we write $-1 = \cos(2n+1)\pi = e^{(2n+1)\pi i}$, where n is any integer; then we have

$$\begin{aligned} -1^{\sqrt{2}} &= \exp\left(\sqrt{2}\,\mathrm{Log}\,(-1)\right) = \exp\left(\sqrt{2}(2n+1)\pi i\right) \\ &= \cos\left(\sqrt{2}(2n+1)\pi\right) + i\sin\left(\sqrt{2}(2n+1)\pi\right). \end{aligned}$$

What does this set of numbers look like? It turns out that this set is actually *infinite*; this is because $\sqrt{2}$ is irrational: if the set were finite, we would have integers $n \neq m$ and k such that

$$\sqrt{2}(2n+1)\pi = \sqrt{2}(2m+1)\pi + 2k\pi,$$

which would give $\sqrt{2} = \frac{k}{n-m}$, contradicting irrationality of $\sqrt{2}$. It is also clear that all of these numbers lie on the unit circle; thus we have an infinite set of numbers on the unit circle, which means that they cannot be 'evenly spaced' in any meaningful sense. (For those who are familiar with the concept of density, we note that this set is in fact *dense* in the unit circle.)

Even more bizarre things happen when we look at *complex* bases. For example, let us consider i^i. Writing $i = \exp i\left(\frac{\pi}{2} + 2n\pi\right)$, we have

$$i^i = \exp\left(i\left[i\left(\frac{\pi}{2} + 2n\pi\right)\right]\right) = \exp\left(-\frac{\pi}{2} - 2n\pi\right),$$

i.e., the number i^i is an infinite sequence of *real* numbers!

(We hasten to note that these examples are more amusing than indicative, and while it is important to keep in mind that exponentials like $z_1^{z_2}$ can be very ill-behaved compared with their real counterparts, this behaviour will not generally concern us in the remainder of the course.)