Summary:
- We wrap up some loose ends from last time.
- We discuss how differentiation rules from elementary calculus can be extended to the current setting.
- We discuss multiple-valued functions and give a brief introduction to the notion of *branch cut*.

5. Harmonic functions. If a function $f'(z)$ has a derivative throughout a region, we say that it is *analytic* in that region.[1] From last time, we know that if we write f as

$$f(x+iy) = P(x,y) + iQ(x,y),$$

then, assuming that P and Q possess continuous first-order partial derivatives, f will be analytic if P and Q satisfy the Cauchy-Riemann equations

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

It turns out that these equations impose a very strong condition on P and Q, namely that they be *harmonic*, i.e., that they satisfy Laplace's equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Assuming that P and Q possess continuous second-order partial derivatives, this can be shown easily as follows:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial Q}{\partial y} + \frac{\partial}{\partial y}\left[-\frac{\partial Q}{\partial x}\right] = \frac{\partial^2 Q}{\partial x \partial y} - \frac{\partial^2 Q}{\partial y \partial x} = 0,$$

since under the above assumption the mixed partial derivatives of Q commute. The calculation for Q is similar and we leave it to the reader as an exercise.

To summarise, then, we have the implication

$$f \text{ analytic} \Longrightarrow \operatorname{Re} f, \operatorname{Im} f \text{ harmonic}.$$

Note that the reverse implication is false, since if P and Q are two harmonic functions there is in general no reason at all to expect them to satisfy the Cauchy-Riemann equations. Note also that for us the term *harmonic* is applied only to real-valued functions of real variables; we do not speak of a function f of a complex variable being harmonic. (We could define *analytic* for functions of a real variable – it is simply that the function have a convergent power series representation – but we have not done so as we shall have no particular need for this concept by itself.)

Harmonic functions are very important in many areas of physics and science, as they can be used to describe temperature distributions, static electric fields, and steady-state fluid flows, for example. We shall see later that one major application of complex variable theory lies in the use of analytic functions *qua* conformal maps to find solutions to Laplace's equation in nontrivial geometries.

Given a harmonic function P, there is a harmonic function Q, unique up to an additive constant, such that $f(x+iy) = P(x,y) + iQ(x,y)$ is analytic. This is discussed in Goursat, §3, and also in §9 below.

6. Differentiation rules. We have already seen one example (at the end of §4 from last time) where a differentiation rule from elementary calculus carried across essentially unchanged to the current setting. It turns out that almost all of the differentiation rules from elementary calculus do also carry over to functions of a complex variable: for example, the product rule and quotient rule do, since the proofs of those two

[1] The word *analytic*, when applied to a real-valued function of a real variable, means that the function can be extended in a power series, i.e., that the Taylor series of the function converges to the function on some interval. We shall show later that, for functions of a complex variable, existence of the derivative throughout an appropriate region allows us to conclude that the function has derivatives of all orders, and that the Taylor series about each point converges to the function on some disc. Thus our terminology is consistent with the real-variable case.

rules work equally well for complex independent variables as they do for real. This means that derivatives of rational functions (quotients of polynomials) can be found exactly as for functions of a real variable.

The chain rule also carries over to the current setting, as can be seen as follows. Suppose that f and g are analytic functions, and let $z \in \operatorname{dom} f$ be such that $f(z) \in \operatorname{dom} g$. Then since f and g are analytic we have

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f'(z), \qquad \lim_{h'\to 0} \frac{g(f(z)+h')-g(f(z))}{h'} = g'(f(z)).$$

Now the first relation can be rewritten in the following way:

$$\lim_{h\to 0} \frac{f(z+h)-f(z)-f'(z)h}{h} = 0.$$

Let us write $\epsilon(h) = f(z+h) - f(z) - f'(z)h$, so that this result becomes $\lim_{h\to 0} \frac{\epsilon(h)}{h} = 0$. Similarly let us write $\epsilon'(h') = g(f(z)+h') - g(f(z)) - g'(f(z))h'$.[2] Then we note that

$$\begin{aligned}
\lim_{h\to 0} \frac{g(f(z+h))-g(f(z))}{h} &= \lim_{h\to 0} \frac{g(f(z)+f'(z)h+\epsilon(h))-g(f(z))}{h} \\
&= \lim_{h\to 0} \frac{g'(f(z))[f'(z)h+\epsilon(h)]+\epsilon'(f'(z)h+\epsilon(h))}{h} \\
&= g'(f(z))f'(z) + \lim_{h\to 0}\left[g'(f(z))\frac{\epsilon(h)}{h} + \frac{\epsilon'(f'(z)h+\epsilon(h))}{h}\right];
\end{aligned}$$

but the limit of the first fraction is zero by what we know about $\epsilon(h)$, while the limit of the second is also zero by what we know about $\epsilon(h)$ and $\epsilon'(h)$. Thus we have

$$\frac{d}{dz} g(f(z)) = g'(f(z))f'(z),$$

exactly as we do in elementary calculus.

We shall see shortly that, given appropriate extensions of the elementary transcendental functions of calculus (the trigonometric, exponential, and logarithmic functions), the derivatives of all of these functions are also what one would expect from calculus.

7. Roots and branch cuts. There is one class of functions which we have already extended to all complex numbers but whose derivatives we have not yet discussed, namely the roots. It turns out that a study of these functions reveals a subtlety in functions of a complex variable which is not visible in functions of a real variable. Let us fix some positive integer m and consider mth roots. Recall that if $z = r(\cos\theta + i\sin\theta)$ is any complex number, then the m complex numbers

$$w_n = r^{1/m}\left(\cos\frac{\theta+2\pi n}{m} + i\sin\frac{\theta+2\pi n}{m}\right)$$

all satisfy $w_n^m = z$. Now a function must have a unique value at a given point; thus if we wish to define an mth root *function* we must have some way of choosing just one of these values for each point. At first sight it would appear that we could just take w_0 and be done, but a bit more thought reveals that the situation is not quite that simple: for example, should $z = r$, for r a positive real number, be represented as

$$z = r(\cos 0 + i\sin 0), \qquad \text{with } m\text{th root} \qquad w_0 = r^{1/m},$$

or as

$$z = r(\cos 2\pi + i\sin 2\pi), \qquad \text{with } m\text{th root} \qquad w_0 = r^{1/m}\left(\cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}\right)?$$

[2] For those who have seen this notation, we note that this is equivalent to saying that $\epsilon(h) = o(h)$ and $\epsilon'(h') = o(h')$.

If we are interested only in defining a function we may just choose one of these and be done. The problem with that method, though, is that the resulting function will not be continuous across the real axis. For suppose that we make the requirement that $\theta \in [0, 2\pi)$, which corresponds to choosing the first of these two expressions. Let us consider the two limits

$$\lim_{h\to 0^+} (\cos h + i \sin h)^{1/m} \qquad \text{and} \qquad \lim_{h\to 0^-} (\cos h + i \sin h)^{1/m}.$$

For our mth root function to be continuous these two limits must be equal. But since we have required the angle θ to lie in the interval $[0, 2\pi)$, we must rewrite the second number as

$$\cos(2\pi + h) + i \sin(2\pi + h)$$

(remember that here h is negative so $2\pi + h < 2\pi$!), which means that the two limits become

$$\begin{aligned}\lim_{h\to 0^+} (\cos h + i \sin h)^{1/m} &= \lim_{h\to 0^+} \left(\cos \frac{h}{m} + i \sin \frac{h}{m}\right)\\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lim_{h\to 0^-} (\cos(2\pi + h) + i \sin(2\pi + h))^{1/m} &= \lim_{h\to 0^-} \left(\cos \frac{2\pi + h}{m} + i \sin \frac{2\pi + h}{m}\right)\\ &= \cos \frac{2\pi}{m} + i \sin \frac{2\pi}{m},\end{aligned}$$

and these two expressions are clearly not equal unless $m = 1$ (when everything is quite trivial). A similar problem would happen if we made the second choice above.

It turns out that the above difficulty is not just a result of our lack of cleverness: there is in fact no way to define an mth root function which is single-valued and continuous on the entire complex plane. The basic idea is already contained in the foregoing. Suppose that $f : \mathbf{C} \to \mathbf{C}$ were a function of a complex variable satisfying everywhere on $\mathbf{C}$ the formula

$$[f(z)]^m = z,$$

and such that $f(z)$ were continuous everywhere on $\mathbf{C}$. Let us consider how f behaves on the unit circle. By our study of roots above, we know that there must be integer $n \in \{0, 1, 2, \cdots, m - 1\}$ such that $f(1) = \cos \frac{2\pi n}{m} + i \sin \frac{2\pi n}{m}$. Since f is continuous, for θ close to zero we must also have

$$f(\cos\theta + i \sin\theta) = \cos \frac{\theta + 2\pi n}{m} + \sin \frac{\theta + 2\pi n}{m}.$$

Now let us consider what happens when we gradually increase θ more and more. Clearly we must always still have

$$f(\cos\theta + i \sin\theta) = \cos \frac{\theta + 2\pi n}{m} + \sin \frac{\theta + 2\pi n}{m},$$

since otherwise there would be a point where we would need to switch to a different value of n, and this would lead to a discontinuity in f (this could be shown analogously to how we argued above about discontinuity across the real axis). Thus we can keep on going up until we get close to 2π. But if θ is very close to 2π the above result gives

$$f(\cos\theta + i \sin\theta) = \cos \frac{\theta + 2\pi n}{m} + \sin \frac{\theta + 2\pi n}{m};$$

but since we can consider $\theta < 0$ as well as $\theta > 0$, we also have

$$f(\cos\theta + i \sin\theta) = f(\cos(\theta - 2\pi) + i \sin(\theta - 2\pi)) = \cos \frac{\theta + 2\pi(n-1)}{m} + \sin \frac{\theta + 2\pi(n-1)}{m},$$

a contradiction.

Let us sum up what we have shown: No matter which choice of mth root we choose, if we continue it along a curve which encloses the origin, it will come back as a different root when we come back to the original point. This phenomenon is actually quite common in the study of functions of a complex variable, and the origin is what is called a *branch point* of the mth root function. Far from being a failure of the theory, it actually leads to very interesting new mathematical structures called *Riemann surfaces*, which we discuss momentarily.

It turns out that if we wish to define an mth root function, there are two distinct ways to proceed. First of all, we could restrict the domain by removing (say) a ray from the origin to infinity from the domain of the function; for example, if we remove the positive real axis together with the origin, it is clear that we may make any single choice of n and get a continuous mth root function on the remaining set. The same is true if we remove any other ray from the origin to infinity. In this setting, the ray we remove from the domain of f is termed a *branch cut*. See Goursat, §6, especially the discussion around Figure 5; see also some additional discussion in §8 herein, below.

Goursat's discussion of cutting the plane relates to the notion of a Riemann surface, which is part of the second possible route out of our difficulties, namely extending the domain to a so-called *m-sheeted cover* of the complex plane.[3] This is rather complicated and we shall only sketch it. The idea is to consider the point 1 on the real axis as distinct from the point obtained by rotating it around the origin once, twice, thrice, $\ldots$, $m-1$ times, but as *the same* as what one gets by rotating m times.[4] This gives m different 'sheets' – in some sense, m different 'copies' of the complex plane – which are joined onto each other in some fashion (think of a spiral staircase which somehow ends up where it started); and we can then define the mth root function by choosing root n on the nth of the sheets.

[3] I don't suppose anyone has studied covering spaces, but in case anyone has, let me just note that this corresponds to the m-sheeted cover of the unit circle by itself. The universal cover of the unit circle will show up when we talk about the logarithm.

[4] If any of you have some familiarity with the notions of particle physics, you may recall that certain elementary particles, such as the electron, are said to have *spin-1/2*, in that they must be turned around *twice* to look the same (a most peculiar property!); that is exactly the same as what is going on here except that for mth roots we must 'turn around' m times to look the same. While it has been too long since I studied the Dirac equation to be sure of myself here, I doubt this is entirely a coincidence, as those of you who have studied the Dirac equation will probably recall that it arises as a *square-root* of the Klein-Gordon equation.