Summary:

- We discuss another geometric interpretation of complex multiplication.
- We then discuss taking powers and roots of complex numbers, and the geometric interpretation of these operations.

We have just observed that multiplying a complex number by another complex number of unit modulus is equivalent to rotating the original complex number by an angle equal to that of the second complex number. It turns out that multiplication by a general complex number can be viewed as the composition of a rotation and an isotropic scaling. Let us see how this works. Suppose that we have two complex numbers,

$$z = r(\cos\theta + i\sin\theta), \qquad w = r'(\cos\theta' + i\sin\theta').$$

Then their product comes out to be (the angular part is exactly analogous to what we saw at the end of the notes of May 5)

$$\begin{aligned} zw &= rr'(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta') \\ &= rr'(\cos\theta\cos\theta' - \sin\theta\sin\theta' + i[\sin\theta\cos\theta' + \cos\theta\sin\theta']) \\ &= rr'[\cos(\theta+\theta') + i\sin(\theta+\theta')]; \end{aligned} \tag{1}$$

in other words, the point corresponding to zw is exactly the point corresponding to z, rotated by θ' and scaled by r'. This is the sense in which multiplication by a complex number is just a rotation and a scaling. (This is related to some of the problems on the review sheet!)

3. Exponentiation. We have seen that the affect of multiplication on the angular part of a complex number is just a rotation. What happens under exponentiation? Let $z = r(\cos\theta + i\sin\theta)$; then we see that, by the formula in (1) above,

$$\begin{aligned} z^2 &= z\cdot z = r^2(\cos 2\theta + i\sin 2\theta), \\ z^3 &= z\cdot z^2 = r(\cos\theta + i\sin\theta)\cdot r^2(\cos 2\theta + i\sin 2\theta) = r^3(\cos 3\theta + i\sin 3\theta), \end{aligned}$$

and so on, so that it is evident that for any positive integer m we have

$$z^m = r^m(\cos m\theta + i\sin m\theta).$$

To try to get some sense of what this means geometrically, let us first consider the case $r = 1$; then $r^m = 1$ for all m and we have simply

$$z^m = \cos m\theta + i\sin m\theta.$$

Now any complex number of unit modulus is represented in the complex plane by a point on the unit circle, and completely determined by the angle between a ray drawn from the origin to that point and the positive x-axis, measured in a counterclockwise direction: this is just the number θ above. This formula then tells us that the point corresponding to z^m is also on the unit circle, but with an angle from the positive x-axis equal to m times that of the point corresponding to z. In other words, if we must traverse an angle θ to arrive at z, we must traverse an angle of $m\theta$ to arrive at z^m.

Suppose now that we consider the affect of exponentiation on not just a single point on the unit circle but rather an *arc*, say from $\theta = 0$ to $\theta = \theta_0$ for some $\theta_0 > 0$. The point corresponding to θ_0, namely $\cos\theta_0 + i\sin\theta_0$, will be mapped by this exponentiation to $\cos m\theta_0 + i\sin m\theta_0$; and it is clear that every point with $\theta \in [0, \theta_0]$ will be mapped to a point with $\theta \in [0, m\theta_0]$. Thus exponentiation simply stretches out the original arc.

With this in mind, let us consider the affect of exponentiation on an angular wedge, namely on the set of all points (of whatever modulus) whose angle with the positive x-axis lies between 0 and θ_0. Such a point can be written in the form $z = r(\cos\theta + i\sin\theta)$, where $\theta \in [0, \theta_0]$, and $z^m = r^m(\cos m\theta + i\sin m\theta)$; from the foregoing, then, it is clear that this point will lie inside a 'wedge' (it may have an angle greater than π and hence not really be a proper 'wedge' anymore) extending from 0 to $m\theta_0$.

Now there is no particular reason to restrict the lower angular bound on the wedge to be 0; we may as well consider a wedge $[\theta_1, \theta_2]$. The same logic shows that this will be mapped to a wedge $[m\theta_1, m\theta_2]$.

In particular, if we consider the wedge from 0 to π and let $m = 2$, we see that the image under exponentiation is the 'wedge' from 0 to 2π, i.e., the entire complex plane. The same is true if we consider the wedge from 0 to $\frac{2\pi}{3}$ and let $m = 3$, and in general, if m is any positive integer, then the wedge from 0 to $\frac{2\pi}{m}$ will be mapped to the entire complex plane by the map $z \mapsto z^m$. Similarly, the wedge from $\frac{2\pi}{m}$ to $\frac{4\pi}{m}$ will also be mapped to the entire complex plane, and so will the wedges from $\frac{2n\pi}{m}$ to $\frac{2(n+1)\pi}{m}$ for any $n = 0, 1, \ldots, m-1$.

While we do not quite have all of the necessary tools to make the following picture precise, it provides much useful intuition and I think is simple enough to understand. We may think of exponentiation by a positive integer as an endpoint in a process that starts with exponentiation by 1 (i.e., doing nothing!) and then slowly increases the exponent *through all real numbers* until it reaches m. Under this kind of a map, the wedge from 0 to $\frac{2\pi}{m}$ (say) will be slowly stretched out (with the bottom edge, i.e., that along the x-axis, remaining fixed) until the outer edge finally reaches the x-axis. Under the same map, the wedge from $\frac{2\pi}{m}$ to $\frac{4\pi}{m}$ will behave slightly differently: the lower edge $\frac{2\pi}{m}$ also moves until it reaches the positive x-axis, while the upper edge $\frac{4\pi}{m}$ moves even faster so that by that point it has travelled one full 2π past the positive x-axis. Similar things can be said about the additional wedges.

What all of this means is that under exponentiation by a positive integer, the wedges $\frac{2\pi n}{m}$ to $\frac{2\pi(n+1)}{m}$ are each rotated and stretched in such a way as to cover the entire complex plane exactly once.[1] This means that each complex number is the image under the exponentiation map of exactly one point from each of these wedges. A little thought shows that this means that each complex number (except 0) has exactly m mth roots.

More precisely, suppose that $z = r(\cos\theta + i\sin\theta)$ is some complex number. Now for each positive real number r there is exactly one positive real number R satisfying $R^m = r$, and we denote this unique positive real mth root by $r^{\frac{1}{m}}$. Given this, for $n = 0, 1, \ldots, m-1$, let $w_n = r^{\frac{1}{m}}\left(\cos\frac{\theta+2\pi n}{m} + i\sin\frac{\theta+2\pi n}{m}\right)$; then clearly

$$\begin{aligned} w_n^m &= \left(r^{\frac{1}{m}}\right)^m (\cos(\theta + 2\pi n) + i\sin(\theta + 2\pi n)) \\ &= r(\cos\theta + i\sin\theta) = z, \end{aligned}$$

so that each of the w_n is an mth root of z. More specifically, if we assume that $\theta \in [0, 2\pi]$, then it is clear that w_n is in the nth of the above wedges. We note that $w_m = w_0$, and in general $w_{n+km} = w_n$ for any positive integer k. It can be shown that the w_n are the only complex mth roots of z, and that z therefore has exactly m distinct mth roots, as claimed. [The proof is not that hard: suppose that $w = r'(\cos\theta' + i\sin\theta')$ is any mth root of z, i.e., that $w^m = z$; this means that

$$r'^m(\cos m\theta' + i\sin m\theta') = r(\cos\theta + i\sin\theta),$$

which means that $r'^m = r$, i.e., $r' = r^{\frac{1}{m}}$, and that there is an integer n such that $m\theta' = \theta + 2n\pi$, which gives $\theta' = \frac{\theta}{m} + \frac{2n\pi}{m}$ for some integer n. Now dividing n by m we can find integers q and r such that $n = qm + r$ and $r \in \{0, 1, 2, \ldots, m-1\}$; thus $\theta' = \frac{\theta}{m} + \frac{2(qm+r)\pi}{m} = \frac{\theta}{m} + 2q\pi + \frac{2r\pi}{m}$ and this w is equal to w_r.]

4. Complex derivatives. Cauchy-Riemann equations In first-year calculus we learned that the derivative of a real-valued function of a single real variable, if it exists, is given by the limit

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}.$$

In multivariable calculus, we learned about taking *partial derivatives*, which are derivatives in a single direction at a time; we couldn't take the derivative 'with respect to a vector' since we had no way of dividing by a vector.[2] Those of you who have seen how derivatives of functions from $\mathbf{R}^n$ to $\mathbf{R}^m$ can be viewed as linear operators between those spaces will still recall that the components of the matrix representations of those operators are still calculated as partial derivatives, i.e., even in that case we reduce back to the case of a single function of a single variable.

[1] Well, *almost* exactly once. To be precise we should only include one of the two edges, restricting the angle to lie in a half-open interval.

[2] This is not *entirely* correct and there is in fact a nice way in which the gradient can be viewed as a derivative $\frac{df}{d\mathbf{r}}$. But that is probably more of a notational shorthand than anything fundamental, unlike what we are about to do with complex numbers.

In complex analysis, though, we can go further since we have a well-defined way of dividing by complex numbers even though they are two-dimensional quantities (at least over $\mathbf{R}$!). Let $f : \mathbf{C} \to \mathbf{C}$ be a complex-valued function of a complex variable, and consider the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$

where now h is allowed to be a *complex* number. Since h is complex, this means that we are taking a two-dimensional limit. As we have learned in multivariable calculus, a two-dimensional limit can only exist if directional limits from different directions exist and are equal (and it may fail to exist even then). Let us consider what the above limit looks like in the two cases where we restrict h to go to zero along the real and imaginary numbers (in terms of the complex plane, this means that h goes to zero along the horizontal and vertical axes, respectively). First, let us write out f explicitly in terms of its real and imaginary parts as (writing $z = x + iy$)

$$f(x + iy) = P(x, y) + iQ(x, y),$$

and assume that all partial derivatives $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ exist. If $h = \Delta x$ is real, the quotient inside the limit becomes

$$\begin{aligned}\frac{f(x+iy+\Delta x) - f(x+iy)}{\Delta x} &= \frac{P(x+\Delta x, y) + iQ(x+\Delta x, y) - [P(x,y) + iQ(x,y)]}{\Delta x} \\ &= \frac{[P(x+\Delta x, y) - P(x,y)] + i[Q(x+\Delta x, y) - Q(x,y)]}{\Delta x}.\end{aligned}$$

Since the partial derivatives $\frac{\partial P}{\partial x}$ and $\frac{\partial Q}{\partial x}$ exist, in the limit as Δx goes to zero this becomes

$$\frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}.$$

This gives the original (two-dimensional) limit along the real axis. To find the limit along the imaginary axis, let $h = i\Delta y$ (note the i!); then we obtain

$$\begin{aligned}\frac{f(x+iy+i\Delta y) - f(x+iy)}{i\Delta y} &= \frac{P(x, y+\Delta y) + iQ(x, y+\Delta y) - [P(x,y) + iQ(x,y)]}{i\Delta y} \\ &= -i\left\{\frac{[P(x, y+\Delta y) - P(x,y)] + i[Q(x, y+\Delta y) - Q(x,y)]}{\Delta y}\right\},\end{aligned}$$

so that since the partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial y}$ exist this becomes

$$-i\left\{\frac{\partial P}{\partial y} + i\frac{\partial Q}{\partial y}\right\} = \frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y}.$$

For the full two-dimensional limit to exist, this must equal the limit along the real axis; thus we must have

$$\frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x},$$

which gives the celebrated *Cauchy-Riemann equations*

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

To sum up: for a function f of a complex variable to have a derivative at a point, its real and imaginary components P and Q must have partial derivatives at that point and those partial derivatives must satisfy the Cauchy-Riemann equations. It can be shown (see Goursat, §3) that if the partial derivatives of P and Q are also continuous at the point in question, then these conditions are sufficient in that f is then guaranteed to have a derivative at that point. Functions whose real and imaginary parts satisfy the Cauchy-Riemann equations but which do not have a derivative shall not concern us much in this course.

When f has a derivative at a certain point, by the foregoing that derivative is given by either of the expressions

$$f'(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y}.$$

Other equivalent expressions can also be derived; see Goursat, §3, equation (2).

Let us consider a specific example of the foregoing.

EXAMPLES. Let us consider a very simple function:

$$f(z) = z^2.$$

To find its real and imaginary parts, let $z = x + iy$; then

$$f(z) = f(x+iy) = (x+iy)^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + i(2xy),$$

whence we see that its real and imaginary parts are, respectively,

$$P(x,y) = x^2 - y^2, \qquad Q(x,y) = 2xy.$$

We leave it as a worthwhile exercise to the reader to show that these do in fact satisfy the Cauchy-Riemann equations. Since they certainly have continuous partial derivatives, we see that f must have a derivative at any point z. The formulas above give this derivative as

$$f'(z) = f'(x+iy) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = 2x + 2iy = 2z.$$

(This should not be a surprise, since we know from real-variable calculus that the derivative of x^2 is $2x$.) In this case, we can also derive this result directly, as follows:

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h\to 0} \frac{z^2 + 2zh + h^2 - z^2}{h} = \lim_{h\to 0} \frac{2zh + h^2}{h} = \lim_{h\to 0} (2z + h) = 2z. \end{aligned}$$

This result turns out to be typical: most of the standard functions we are familiar with from calculus which have derivatives as functions of a real variable *also* have derivatives as functions of a *complex* variable, and the derivatives are the same. (There is a very good reason for this, which will become clearer throughout the course: it is tied up with the fact that most of the functions we deal with in calculus do not just have a single derivative but are rather *real analytic*, i.e., are equal to their Taylor series expansions. Such functions always extend to differentiable functions of a complex variable, and this is one of the major links from real to complex variable theory.)

As a still elementary but slightly more complicated example, let us show that the power rule of elementary calculus holds for functions of a complex variable, if we restrict ourselves to positive integer exponents. (It holds for more general exponents, too, at least away from $z = 0$, but that will require a separate treatment.) Thus let m be a positive integer, and define $f(z) = z^m$. Then we have

$$\begin{aligned} \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \frac{(z+h)^m - z^m}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \left(\sum_{k=0}^{m} \binom{m}{k} z^{m-k} h^k - z^m \right) \\ &= \lim_{h\to 0} \frac{1}{h} \left(m z^{m-1} h + \frac{m(m-1)}{2} z^{m-2} h^2 + \cdots \right) \\ &= \lim_{h\to 0} \left(m z^{m-1} + \frac{m(m-1)}{2} z^{m-2} h + \cdots \right) = m z^{m-1}, \end{aligned}$$

since all terms in $\cdots$ have at least an h^2 in them and hence must go to zero as h does. Thus we have $f'(z) = m z^{m-1}$, exactly as in the real-variable case.