Summary:
- We give a description of the complex number system.
- We then give a description of the complex plane and indicate why it is something which might be useful.

# I. INTRODUCTION TO THE COMPLEX PLANE

**1. Complex numbers.** We probably saw complex numbers for the first time when we learned how to solve quadratic equations. For example, the equation

$$x^2 = -1$$

has no solution over the real numbers. It turns out to be useful in algebra, and even more in analysis, to *extend* our number system by including an extra quantity, written $i$, which behaves exactly like a real number except that it has the property

$$i^2 = -1. \tag{1}$$

A general number in our new number system can be written in the form $a + bi$, where $a$ and $b$ are arbitrary *real* numbers,[1] and we require that these numbers satisfy all of the standard rules of algebra, augmented by equation (1). Thus, for example, the product of two complex numbers is given by

$$(a + bi)(c + di) = ac + bi \cdot c + a \cdot di + bi \cdot di = ac + bci + adi + bdi^2 = ac - bd + (bc + ad)i.$$

(As shown here, whenever we write out a complex number we always combine real and imaginary terms when possible.)

We generally use the letters $z$ and $w$ to denote complex numbers, and $x$ and $y$ to denote real numbers. We let $\mathbf{C}$ denote the set of all complex numbers. If $z = a + bi$ is a complex number, we call $a$ the *real part* of $z$ and $b$ the *imaginary part* of $z$, and write $a = \operatorname{Re} z$, $b = \operatorname{Im} z$. Two complex numbers $a + bi$ and $c + di$ are *equal* if and only if their real and imaginary parts are equal.[2]

To every complex number $a+bi$ there corresponds another complex number known as its *conjugate* and given by $a - bi$.[3] If $z$ is any complex number, we write $\overline{z}$ for its conjugate. The conjugate will be seen later to have many uses, but for the moment we note its use in finding inverses. First, note that if $z = a+bi$, then

$$z\overline{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

Thus if $a + bi \neq 0$, then

$$\frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2},$$

which is defined since $a + bi \neq 0$ implies that at least one of $a$ and $b$ is nonzero, so $a^2 + b^2 > 0$. This is the desired formula for the inverse of a complex number.

See Goursat, §1.

**2. The complex plane.** If $z$ is any complex number, it determines two real numbers $\operatorname{Re} z$ and $\operatorname{Im} z$, and is in turn uniquely determined by these two numbers. This suggests that, just as we may think of arbitrary real numbers as points on the *real number line*, we may think of arbitrary complex numbers as points in the *complex plane*. Specifically, given a plane with perpendicular axes which we call $x$ and $y$, we associate with any complex number $z$ the point in this plane whose $x$-coordinate is $\operatorname{Re} z$ and whose $y$-coordinate is $\operatorname{Im} z$. While complex numbers are *per se* abstract objects without any direct concrete significance, this association allows us to think and speak of them as points in the plane. We shall do this whenever it seems

[1] Whenever we write an arbitrary complex number as $a + bi$, it will always be assumed that $a$ and $b$ are real.

[2] This means that the set $\{1, i\}$ is a basis for $\mathbf{C}$ considered as a real vector space.

[3] We remind the reader that the presence of a $+$ or $-$ in front of a quantity does not guarantee the resulting sign; in other words, $+b$ can be negative and $-b$ can be positive, and both will be respectively when $b$ is negative.

convenient; thus we shall speak of "the point $a + bi$", etc., when more carefully we should say "the point corresponding to the complex number $a + bi$".

Given the foregoing, it is clear that the point corresponding to the conjugate of a complex number $a+bi$ is simply the reflection in the $x$-axis of the point corresponding to $a + bi$.

The foregoing connection between complex numbers and points in a plane, while it may be interesting, would not be particularly useful if the *geometric* properties inherent in the Euclidean plane were not somehow related to *algebraic* or *analytic* properties of the complex numbers its points represent. We shall see throughout this course that there are in fact many and deep connections between the geometry of the plane on the one hand and the algebraic and analytic properties of complex numbers on the other. Here we shall indicate one example.

EXAMPLES. One simple example is as follows. Suppose that $z = a + bi$ and $w = c + di$ are any two complex numbers. Then clearly

$$\overline{z}w = (a - bi)(c + di) = ac + bd + i(ad - bc).$$

Now if we think of the vectors (corresponding to the points) corresponding to $a+bi$ and $c+di$, i.e., $\mathbf{v} = a\mathbf{i}+b\mathbf{j}$, $\mathbf{u} = c\mathbf{i}+d\mathbf{j}$, we see that their dot product is $\mathbf{v} \bullet \mathbf{u} = ac+bd$ while their cross product is $\mathbf{v} \times \mathbf{u} = (ad-bc)\mathbf{k}$; in other words, roughly, the real part of $\overline{z}w$ is the dot product of the vectors corresponding to $z$ and $w$, while the imaginary part is their cross product.[4] We shall see some other relations of this sort when we talk about derivatives of functions of a complex variable; it turns out that, when viewed as a vector field, the derivative of the conjugate of such a function essentially encodes the divergence and curl of the vector field.[5]

As another, more interesting, example, let $a+bi$ be any complex number, and consider the corresponding point in the plane. This point has polar coordinates $(r, \theta)$, where $r$ is the distance from the origin to the point and $\theta$ is the angle from the positive $x$-axis to the ray from the origin passing through the point. In symbols, this becomes

$$r = \sqrt{a^2 + b^2}, \qquad \cos\theta = \frac{a}{\sqrt{a^2 + b^2}}, \qquad \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}$$
$$a = r\cos\theta, b = r\sin\theta$$

Note that $\theta$ is only defined up to a multiple of $2\pi$: the two polar coordinate expressions $(r, \theta)$ and $(r, \theta + 2\pi)$ determine exactly the same point in the plane. We shall see shortly that for many important functions to be continuous (in an appropriate sense) on the complex plane, there is no way around this ambiguity: it is simply something which must be dealt with.

Now suppose that $c + di$ is any other complex number which satisfies $c^2 + d^2 = 1$: this means that the point corresponding to $c + di$ lies on the unit circle. If we let $(r_0, \theta_0)$ denote the polar coordinates of this point, then we have $r_0 = 1$, while $\theta_0$ satisfies $\cos\theta_0 = c$, $\sin\theta_0 = d$.[6] Now applying basic trigonometric identities, we obtain

$$\begin{aligned}
(a + bi)(c + di) &= ac - bd + i(ad + bc) \\
&= r\cos\theta\cos\theta_0 - r\sin\theta\sin\theta_0 + i(r\cos\theta\sin\theta_0 + r\sin\theta\cos\theta_0) \\
&= r\cos(\theta + \theta_0) + ir\sin(\theta + \theta_0) \\
&= r\left[\cos(\theta + \theta_0) + i\sin(\theta + \theta_0)\right],
\end{aligned}$$

from which it is evident that the point corresponding to the product $(a + bi)(c + di)$ is simply that corresponding to $a + bi$ rotated counterclockwise by the angle $\theta_0$!

[4] It turns out that there is a *four-dimensional* extension of the real numbers called the *quaternions*, which contain the complex numbers, and which in some sense generalises results of this sort to full three-dimensional vectors. We shall not deal with these in this course, though, except for a few asides like this one.

[5] While interesting, these examples are somewhat tangential to the main content of this course.

[6] If you are familiar with De Moivre's theorem, it is useful to note that this means that $c + di = \cos\theta_0 + i\sin\theta_0$.