MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR AUGUST 10 – 14

Due Friday, August 14, at 11:59 PM EDT.

1. [15 marks; modified version of Question 3 from last week's assignment.] Suppose that f is a nonzero function which is analytic on the entire complex plane. ('Nonzero' here means that there is some point in the complex plane at which f is not zero. It does not mean that f has no zeros in the plane.) Let C_R denote the (full) circle of radius R centred at the origin. Is it possible to have

$$\lim_{R\to\infty}\int_{C_R}|f(z)|\,ds = 0?$$

(The integral here is an arclength integral from multivariable calculus.) If not, prove it; otherwise, give an example. [Hint: check the proof of Liouville's Theorem (the one in the lecture notes)!]

We may proceed as in the proof of Liouville's Theorem in the lecture notes. Let $z_0 \in \mathbf{C}$[1 mark], and let $R > |z_0| + 1$[2 marks]. By the Cauchy integral formula, we have, letting C_R denote the circle of radius R centred at the origin[1 mark],

$$|f(z_0)| = \left|\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-z_0}\,dz\right| \text{[1 mark]} \leq \frac{1}{2\pi}\int_{C_R}\left|\frac{f(z)}{z-z_0}\right|\,ds\text{[2 marks]} \leq \frac{1}{2\pi}\int_{C_R}|f(z)|\,ds\text{[1 mark]},$$

since if z is on C_R then $|z - z_0| \geq 1$[2 marks]. But if we now take R to infinity, this last integral must vanish[1 mark], giving $|f(z_0)| = 0$[1 mark], or $f(z_0) = 0$[1 mark]. But $z_0 \in \mathbf{C}$ was arbitrary[1 mark], so f must be identically zero[1 mark].

[Marking: as above. Another method would be to show that all of the coefficients of the Taylor series of f vanish at the origin – this would be much closer to the proof of Liouville's Theorem in Goursat. On the other hand, the vanishing of the integral does *not* imply directly that the modulus of f must be bounded – if you said that you may have gotten very few marks.]

2. [10 marks] (a) Show that for all $z = x + iy \in \mathbf{C}$,

$$|\sin z| \leq e^{|y|}.$$

Let $z = x + iy$; then

$$|\sin z| = \left|\frac{e^{iz}-e^{-iz}}{2i}\right| \text{[1 mark]} \leq \frac{1}{2}\left(\left|e^{ix+y}\right| + \left|e^{-ix+y}\right|\right) \text{[1 mark]} \leq \frac{1}{2}\left(e^{-y}+e^{y}\right) \text{[1 mark]} \leq e^{|y|}\text{[1 mark]}.$$

(b) Using part (a) and Rouché's Theorem, determine how many zeros the function

$$3z^8 + \sin z$$

has in the unit disk. [It is worth spending some time thinking about whether the same procedure can be applied on arbitrarily large disks. But you do not need to say anything about that in your solution.]

If $z = x + iy$ is any point on the unit circle[1 mark], we ahve $|y| \leq 1$[1 mark], so

$$\left|3z^8\right| = 3 \geq e \geq e^{|y|} \geq |\sin z|, \text{[1 mark]}$$

and by Roché's Theorem the function $3z^8 + \sin z$ must have as many zeroes as $3z^8$[1 mark] (counting multiplicities[1 mark]), i.e., 8.[1 mark]

[Marking: as above.]

3. [10 marks] (a) Find a polynomial solution to the following problem on the unit disk $D = \{z \,|\, |z| < 1\}$:

$$\Delta u = 0, \qquad u|_{\partial D} = \cos\theta,$$

where θ is the usual polar coordinate on the plane.

We look for a linear solution of the form $u = a + bx + cy$[1 mark]. On the unit circle, we have $x = \cos\theta$, $y = \sin\theta$[1 mark]; thus we require

$$a + b\cos\theta + c\sin\theta = \cos\theta, \text{[1 mark]}$$

from which we see that $a = c = 0$, $b = 1$[1 mark]. Thus $u = x$ will solve the given problem.[1 mark]

(b) Use your solution to (a) and a conformal transformation to solve the following problem on the *exterior* of the unit disk, i.e., the set $E = \{z \,|\, |z| > 1\}$:

$$\Delta u = 0, \qquad u|_{\partial E} = \cos\theta, \qquad u \to 0 \text{ as } |z| \to \infty.$$

(Note that $\partial E = \partial D$, both being just the unit circle.)

The function $z \mapsto 1/z$ will take the interior of the unit circle (without the origin) to the exterior[1 mark]; thus the function (setting $z = x + iy$)

$$v(x, y) = u(1/z) = u\left(\frac{x - iy}{x^2 + y^2}\right) = \frac{x}{x^2 + y^2}\text{[1 mark]}$$

will be harmonic on the exterior of the unit disk[1 mark]. Since on the unit disk it satisfies

$$v = \frac{\cos\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta, \text{[1 mark]}$$

and if $z = R(\cos\theta + i\sin\theta)$, $R > 0$, it is

$$v = \frac{R\cos\theta}{R^2} = \frac{1}{R}\cos\theta\text{[1 mark]}$$

which goes to zero as $|R| = z \to \infty$, this v must be the solution to our problem.
[Marking: as above.]

4. [15 marks] Solve the following problem on the lower half-plane $H = \{x + iy \,|\, y < 0\}$:

$$\Delta u = 0, \qquad u|_{\partial H} = \begin{cases} \pi, & x < -1 \\ \cos^{-1} x, & x \in (-1, 1) \\ 0, & x > 1 \end{cases}$$

(Note that ∂H, the boundary of H, is just the real axis.)

We transform this problem using the map $z \mapsto \cos z$[1 mark] from the rectangle $R = \{x + iy \,|\, x \in (0, \pi),\ y > 0\}$[1 mark] to the lower half-plane. We must determine the new boundary data. The line $x = 0$, $y > 0$ is mapped to the interval $(1, +\infty)$, on which u is 0; thus the transformed data will also be 0 there[1 mark]. Similarly, the line $x = \pi$, $y > 0$ is mapped to the segment $(-\infty, -1)$, on which u is π; thus the transformed ata must be 1 there[1 mark]. Finally, on the segment $y = 0$, $x \in (0, \pi)$, we have

$$\cos z = \cos x,$$

i.e., the point $(x, 0)$ is mapped to $(\cos x, 0)$[1 mark], so

$$u(\cos z) = u(\cos x, 0) = \arccos\cos x = x\text{[1 mark]}$$

since we are using the branch of arccos which takes $[-1, 1]$ into $[0, \pi]$, and $x \in (0, \pi)$. Thus the transformed problem is

$$\Delta v = 0 \text{ on } R, \qquad \mathbf{v}|_{\partial R} = \begin{cases} 0, & x = 0 \\ x, & y = 0 \\ \pi, & x = \pi \end{cases}.\text{[1 mark]}$$

We seek a linear solution to this problem; thus we write $v = a + bx + cy$[1 mark], and try to solve for a, b, and c, using the boundary conditions. These give

$$\begin{aligned} a + cy &= 0 \\ a + bx &= x \qquad \text{[2 marks]} \\ a + b\pi + cy &= \pi; \end{aligned}$$

ince these must hold for all x and y, we have $a = c = 0$, $b = 1$[2 marks], so $v = x$[1 mark]. More carefully, we have $v(x, y) = x$, or $v(x + iy) = x$, so $v(z) = \operatorname{Re} z$. Thus $u(z) = (v \circ \arccos)(z) = \operatorname{Re} \arccos z$[1 mark]. We may express this in terms of $\operatorname{Re} \arcsin z$ as follows[1 mark]. We have the formula

$$\sin(w + \pi/2) = \cos w;$$

thus if $w = \arccos z$ we have

$$z = \cos w = \sin(w + \pi/2),$$

so $w = \arcsin z - \pi/2$, at least if we take the right branch of arcsin. Now the branch of arccos we use maps into $[0, \pi]$, while the branch of arcsin used in the notes maps into $[-\pi/2, \pi/2]$; thus we must replace arcsin by $\pi - \arcsin$, meaning that we have finally the solution

$$u(z) = \frac{\pi}{2} - \operatorname{Re} \arcsin z.$$

[Marking: as above. This exercise is quite close to the example on pp. 3 – 6 of the August 13 lecture notes on the course website.]