MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR AUGUST 10 – 14

Due Friday, August 14, at 11:59 PM EDT.

1. [15 marks; modified version of Question 3 from last week's assignment.] Suppose that f is a nonzero function which is analytic on the entire complex plane. ('Nonzero' here means that there is some point in the complex plane at which f is not zero. It does not mean that f has no zeros in the plane.) Let C_R denote the (full) circle of radius R centred at the origin. Is it possible to have

$$\lim_{R\to\infty}\int_{C_R}|f(z)|\,ds = 0?$$

(The integral here is an arclength integral from multivariable calculus.) If not, prove it; otherwise, give an example. [Hint: check the proof of Liouville's Theorem (the one in the lecture notes)!]

2. [10 marks] (a) Show that for all $z = x + iy \in \mathbf{C}$,

$$|\sin z| \le e^{|y|}.$$

(b) Using part (a) and Rouché's Theorem, determine how many zeros the function

$$3z^8 + \sin z$$

has in the unit disk. [It is worth spending some time thinking about whether the same procedure can be applied on arbitrarily large disks. But you do not need to say anything about that in your solution.]

3. [10 marks] (a) Find a polynomial solution to the following problem on the unit disk $D = \{z \,|\, |z| < 1\}$:

$$\Delta u = 0, \qquad u|_{\partial D} = \cos\theta,$$

where θ is the usual polar coordinate on the plane.

(b) Use your solution to (a) and a conformal transformation to solve the following problem on the *exterior* of the unit disk, i.e., the set $E = \{z \,|\, |z| > 1\}$:

$$\Delta u = 0, \qquad u|_{\partial E} = \cos\theta, \qquad u \to 0 \text{ as } |z| \to \infty.$$

(Note that $\partial E = \partial D$, both being just the unit circle.)

4. [15 marks] Solve the following problem on the lower half-plane $H = \{x + iy \,|\, y < 0\}$:

$$\Delta u = 0, \qquad u|_{\partial H} = \begin{cases} \pi, & x < -1 \\ \cos^{-1} x, & x \in (-1, 1) \\ 0, & x > 1 \end{cases}$$

(Note that ∂H, the boundary of H, is just the real axis.)