MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR AUGUST 3 – 7

Due Monday, August 10, at 11:59 PM EDT.

1. [15 marks] Evaluate whichever of the integrals from problem 3 of the last assignment you did not do last week.

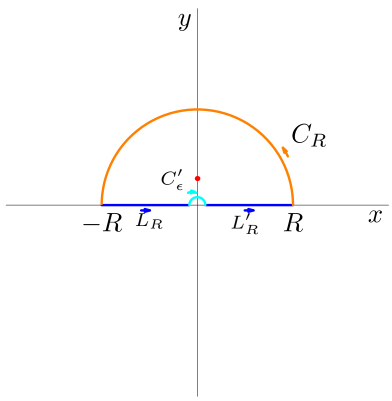
[See the previous set of solutions! – also for marking scheme!]

2. [20 marks] Evaluate the following integral:

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx,$$

where $\log x$ denotes the usual real-valued logarithm of the positive real number x.

Our first inclination (well, Nathan's first inclination, anyway!) may be to use a keyhole contour like that used for the second integral in 1. However, this won't work, as the portions of the integrals on the line segments across the branch containing $\log x/(1+x^2)^2$ will cancel. Thus we instead use the following contour,[2 marks] which is almost the same as the one we used when evaluating integrals involving $\sin x/x$ except that we break it up into more pieces. As in the second integral in 1, in order to integrate over contours in the complex plane we must choose a particular branch of the logarithm.[1 mark]In this case it turns out to be convenient to take a branch cut which is far away from the contour; thus we shall take a branch cut along the negative imaginary axis, and require our angle to lie in the interval $(-\pi/2, 3\pi/2)$. [A branch cut along any line in the lower half-plane would work equally well, and in fact the calculations below would be unchanged. While one could also use a branch cut along part of the real axis, it would in this case only lead to additional complications.]



Now if z is any point on C_R, then we may write $z = Re^{i\theta}$, where $\theta \in [0,\pi]$; thus $\operatorname{Log} z = \log R + i\theta$, so $|\operatorname{Log} z| \le |\log R| + \pi$ and we may write

$$R\left|\frac{\operatorname{Log} z}{(1+z^2)^2}\right| \le R\frac{|\log R|+\pi}{(R^2-1)^2},\text{[1 mark]}$$

which clearly goes to zero as $R \to \infty$ (remember that $\log R < R$ for all R), so

$$\lim_{R\to\infty}\int_{C_R}\frac{\operatorname{Log} z}{(1+z^2)^2}\,dz = 0.\text{[1 mark]}$$

Similarly, if z is any point in C'_ϵ, then we may write $z = \epsilon e^{i\theta}$, where again $\theta \in [0,\pi]$, so as before we have $|\operatorname{Log} z| \le |\log \epsilon| + \pi$ and

$$\epsilon\left|\frac{\operatorname{Log} z}{(1+z^2)^2}\right| \le \epsilon\frac{|\log\epsilon|+\pi}{(1-\epsilon^2)^2}.\text{[1 mark]}$$

Now clearly

$$\lim_{\epsilon\to 0^+}\epsilon\frac{\pi}{(1-\epsilon^2)^2} = 0;\text{[1 mark]}$$

also, by L'Hôpital's rule (for real functions!),

$$\lim_{\epsilon\to 0^+} \epsilon \log \epsilon = \lim_{\epsilon\to 0^+} \frac{\log \epsilon}{1/\epsilon} = \lim_{\epsilon\to 0^+} \frac{1/\epsilon}{-1/\epsilon^2} = \lim_{\epsilon\to 0^+} -\epsilon = 0, \text{[1 mark]}$$

from which we see that

$$\lim_{\epsilon\to 0^+} \int_{C'_\epsilon} \frac{\operatorname{Log} z}{(1+z^2)^2}\, dz = 0.$$

Thus, in the limits $\epsilon \to 0^+$ and $R \to \infty$, we have, since the only singularity of the integrand inside the contour is a pole at $z = i$,

$$\lim_{\epsilon\to 0^+} \lim_{R\to\infty} \int_{L_1} \frac{\operatorname{Log} z}{(1+z^2)^2}\, dz + \int_{L_2} \frac{\operatorname{Log} z}{(1+z^2)^2}\, dz = 2\pi i \operatorname{Res}_i \frac{\operatorname{Log} z}{(1+z^2)^2}. \text{[1 mark]}$$

Before computing the residue, we note that $-L_1$ may be parameterised as $te^{i\pi}$, $t \in [0, R]$, so that

$$\int_{L_1} \frac{\operatorname{Log} z}{(1+z^2)^2}\, dz = \int_0^R \frac{\log t + i\pi}{(1+t^2)^2}\, dt = \int_{L_2} \frac{\operatorname{Log} z}{(1+z^2)^2}\, dz + i\pi \int_0^R \frac{1}{(1+t^2)^2}\, dt. \text{[2 marks]}$$

Thus we have also to evaluate the integral

$$\int_0^\infty \frac{1}{(1+t^2)^2}\, dt = \frac{1}{2} \int_{-\infty}^\infty \frac{1}{(1+t^2)^2}\, dt;$$

this latter integral may be evaluated by closing the contour in the upper half-plane,[1 mark]noting that the integral over the extra semicircle will go to zero as its radius goes to infinity since

$$R \left| \frac{1}{(1+R^2 e^{2it})^2} \right| \leq R \frac{1}{(R^2-1)^2} \to 0 \quad \text{as} \quad R \to \infty, \text{[1 mark]}$$

and thus obtaining

$$\int_{-\infty}^\infty \frac{1}{(1+t^2)^2}\, dt = 2\pi i \operatorname{Res}_i \frac{1}{(1+z^2)^2} \text{[1 mark]} = 2\pi i \left. \frac{d}{dz} \frac{1}{(z+i)^2} \right|_{z=i} = 2\pi i \left(-\frac{2}{(2i)^3} \right) = \frac{\pi}{2}, \text{[2 marks]}$$

so finally

$$\int_0^\infty \frac{1}{(1+t^2)^2}\, dt = \frac{\pi}{4}$$

and

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\, dx = \frac{1}{2} \left[2\pi i \operatorname{Res}_i \frac{\operatorname{Log} z}{(1+z^2)^2} - i\frac{\pi^2}{4} \right].$$

Thus we need only compute this residue. We have

$$\operatorname{Res}_i \frac{\operatorname{Log} z}{(1+z^2)^2} = \left. \frac{d}{dz} \frac{\operatorname{Log} z}{(z+i)^2} \right|_{z=i} = \frac{\frac{1}{i}(2i)^2 - 2(2i)\operatorname{Log} i}{(2i)^4} = \frac{-\frac{4}{i} - 4i\frac{i\pi}{2}}{16} = \frac{i}{4} + \frac{\pi}{8}, \text{[3 marks]}$$

whence finally

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\, dx = \frac{1}{2} \left[-\frac{\pi}{2} + i\frac{\pi^2}{4} - i\frac{\pi^2}{4} \right] = -\frac{\pi}{4}. \text{[1 mark]}$$

We note that it is reasonable to obtain a negative number since $\log x \to -\infty$ as $x \to 0^+$.
[Marking: as above.]