MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR JULY 20 – 31

Due Saturday, August 1, at 10:00 PM EDT.

1. Consider the integral from question 2 of the previous homework assignment:

$$\int_{-\infty}^{+\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx,$$

and assume that both m and a are positive real numbers. By using an indented contour, evaluate this integral fully. [You are allowed to resubmit material submitted as part of the previous assignment if you wish.]

By $(\sin mx)/x$ in the integrand is meant the function

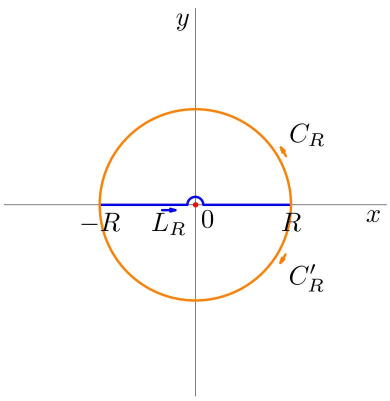
$$\begin{cases} \frac{\sin mx}{x}, & x \neq 0 \\ m, & x = 0, \end{cases}$$

which is just $mf(mx)$ if we set

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0, \end{cases}$$

which as we have seen already extends to an analytic function on the entire complex plane. By the Cauchy integral theorem, then, we can replace the integral over the real axis with an integral over an indented contour, as in the figure. We denote this by L_R, the upper semicircle by C_R, and the lower semicircle by C'_R, as shown in the figure. Now since the contour L_R does not pass through the origin, we may write

$$\begin{aligned}\int_{-R}^{R} \frac{\sin mx}{x(x^2+a^2)}\,dx &= \int_{L_R} \frac{\sin mz}{z(z^2+a^2)}\,dz \\ &= \int_{L_R} \frac{e^{imz}-e^{-imz}}{2iz(z^2+a^2)}\,dz = \int_{L_R} \frac{e^{imz}}{2iz(z^2+a^2)}\,dz - \int_{L_R} \frac{e^{-imz}}{2iz(z^2+m^2)}\,dz,\end{aligned}$$



and we may evaluate these integrals by closing in the upper and lower half-planes, respectively, and applying Jordan's lemma. Specifically, since $m > 0$ and on C_R and C'_R we have

$$\lim_{R\to\infty} \left| \frac{1}{2iz(z^2+a^2)} \right| \leq \lim_{R\to\infty} \frac{1}{2R(R^2-a^2)} = 0,$$

Jordan's lemma gives

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{imz}}{2iz(z^2+a^2)}\,dz = 0$$

and so, since ia is the only pole of the integrand in the contour $L_R + C_R$,

$$\lim_{R\to\infty}\int_{L_R}\frac{e^{imz}}{2iz(z^2+a^2)}\,dz = 2\pi i \mathrm{Res}_{ia}\frac{e^{imz}}{2iz(z^2+a^2)}$$
$$= 2\pi i \lim_{z\to ia}(z-ia)\frac{e^{imz}}{2iz(z-ia)(z+ia)}$$
$$= 2\pi i\frac{e^{-ma}}{-2\pi(2ia)} = -\frac{e^{-ma}}{2a^2}\pi.$$

Similarly, since $m > 0$, on the lower half-circle we have also by the modified form of Jordan's lemma we used once before

$$\lim_{R\to\infty}\int_{C_R'}\frac{e^{-imz}}{2iz(z^2+a^2)}\,dz = 0$$

and so, since the curve $L_R + C_R'$ is oriented *clockwise* and contains the two poles ia and 0,

$$\lim_{R\to\infty}\int_{L_R}\frac{e^{-imz}}{2iz(z^2+a^2)}\,dz = -2\pi i\left[\mathrm{Res}_{-ia}\frac{e^{-imz}}{2iz(z^2+a^2)} + \mathrm{Res}_0\frac{e^{-imz}}{2iz(z^2+a^2)}\right]$$
$$= -2\pi i\left[\lim_{z\to -ia}(z+ia)\frac{e^{-imz}}{2iz(z-ia)(z+ia)} + \lim_{z\to 0} z\frac{e^{-imz}}{2iz(z^2+a^2)}\right]$$
$$= -2\pi i\left[\frac{e^{-ma}}{2a(-2ia)} + \frac{1}{2ia^2}\right] = \pi\frac{e^{-ma}}{2a^2} - \frac{\pi}{a^2},$$

from which we obtain finally

$$\int_{-\infty}^{\infty}\frac{\sin mx}{x(x^2+a^2)} = -\frac{e^{-ma}}{2a^2}\pi - \left[\pi\frac{e^{-ma}}{2a^2} - \frac{\pi}{a^2}\right] = \frac{\pi}{a^2}\left(1 - e^{-ma}\right).$$

[Marking: 1 mark for using an indented contour; 1 mark for closing in both half-planes; in each half-plane, 1 mark for applying Jordan's lemma and 1 mark for the corresponding limit; in each half-plane, 1 mark for the application of the residue theorem; 2 marks for each of three residue calculations; 1 mark for the calculations giving the final answer. 15 marks total.]

2. Evaluate the following integral:

$$\int_0^{2\pi}\frac{dt}{(c-\cos 2t)^2}, \tag{1}$$

where c is a real number with absolute value greater than 1.

This integral becomes far simpler if we first make the substitution $x = 2t$ [1 mark]; the integral then becomes

$$\int_0^{4\pi}\frac{\frac{1}{2}dx}{(c-\cos x)^2} = \int_0^{2\pi}\frac{dx}{(c-\cos x)^2},$$

where the second equality follows since cos is periodic with period 2π. Now we use the usual procedure of viewing this integral as the integral over the unit circle of some function to be determined. We have

$$\cos t = \frac{1}{2}\left(e^{it} + e^{-it}\right),$$

which will equal

$$\frac{1}{2}\left(z + z^{-1}\right) \text{ [2 marks]}$$

if $z = e^{it}$. Now given this z, we have $dz = ie^{it}dt = izdt$; since z is never zero, we may rewrite this as $dt = dz/(iz)$, and rewrite the original integral (1) as (letting C denote the unit circle)

$$\int_C\frac{dz/(iz)}{\left(c - \frac{1}{2}(z+z^{-1})\right)^2} = \frac{1}{i}\int_C\frac{4z\,dz}{(z^2-2cz+1)^2}.\text{[2 marks]}$$

Note that the integrand has two poles, both of second order;[1 mark]they are at the zeroes of $z^2 - 2cz + 1$. These may be found by using the quadratic formula; noting that the discriminant is $4c^2 - 4 = 4(c^2 - 1) > 0$, we may write these as

$$z = c \pm \sqrt{c^2 - 1}.\text{[2 marks]}$$

Now if $c > 1$, then clearly $c + \sqrt{c^2 - 1} > 1$, so only $c - \sqrt{c^2 - 1}$ lies inside the unit circle; while if $c < -1$, then clearly $c - \sqrt{c^2 - 1} < -1$, so only $c + \sqrt{c^2 - 1}$ lies inside the unit circle[2 marks]. Thus we consider these two cases separately. If $c > 1$, we have[4 marks, 1 for the residue theorem, 3 for the residue computation]

$$\begin{aligned}\frac{1}{i}\int_C \frac{4z\,dz}{(z^2 - 2cz + 1)^2} &= 8\pi \mathrm{Res}_{c-\sqrt{c^2-1}} \frac{z}{(z^2 - 2z + 1)^2} \\ &= 8\pi \left.\frac{d}{dz}\frac{z}{(z - c - \sqrt{c^2-1})^2}\right|_{c-\sqrt{c^2-1}} = 8\pi\left[\frac{1}{c^2-1} + \frac{2(c - \sqrt{c^2-1})}{(c^2-1)^{3/2}}\right] \\ &= 8\pi \frac{2c - \sqrt{c^2-1}}{(c^2-1)^{3/2}},\end{aligned}$$

while if $c < 1$, we have[4 marks, 1 for the residue theorem, 3 for the residue computation]

$$\begin{aligned}\frac{1}{i}\int_C \frac{4z\,dz}{(z^2 - 2cz + 1)^2} &= 8\pi \mathrm{Res}_{c+\sqrt{c^2-1}} \frac{z}{(z^2 - 2cz + 1)^2} \\ &= 8\pi \left.\frac{d}{dz}\frac{z}{(z - c + \sqrt{c^2-1})^2}\right|_{c+\sqrt{c^2-1}} = 8\pi\left[\frac{1}{c^2-1} - \frac{2(c + \sqrt{c^2-1})}{(c^2-1)^{3/2}}\right] \\ &= 8\pi \cdot \frac{-2c - \sqrt{c^2-1}}{(c^2-1)^{3/2}},\end{aligned}$$

and we see finally that

$$\int_0^{2\pi} \frac{dt}{(c - \cos 2t)^2} = \frac{16\pi|c|}{(c^2-1)^{3/2}} - \frac{8\pi}{c^2-1}.\text{[2 marks]}$$

[Marking: as indicated.]

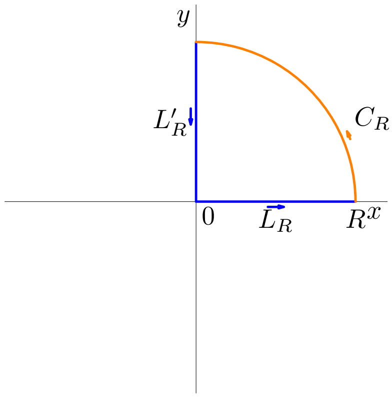
3. Choose *one* of the following integrals, and evaluate it:

$$\int_0^\infty \frac{\cos x^2 - \sin x^2}{x^8 + 1}\,dx, \qquad \int_0^\infty \frac{x^{1/2}}{x^2 + 1}.$$

You are strongly encouraged to also do the other integral for practice! [Hint for the first integral: try evaluating $\int_0^\infty \frac{e^{ix^2}}{x^8+1}\,dx$.]

For the first integral, we close using the quarter-circle contour shown in the figure. Note that, parameterising the circular arc as Re^{it}, $t \in [0, \pi/2]$, we have

$$\begin{aligned}\left|\int_{C_R} \frac{e^{iz^2}}{z^8+1}\,dz\right| &\le \int_0^{\pi/2} R\left|\frac{e^{iR^2\cos 2t}e^{-R^2\sin 2t}}{z^8+1}\right|\,dt \\ &\le \frac{R}{R^8-1}\frac{1}{2}\int_0^\pi e^{-R^2\sin x}\,dx = \frac{R}{R^8-1}\int_0^{\pi/2} e^{-R^2\sin x}\,dx \\ &= \frac{R}{R^8-1}\frac{\pi}{2R^2}\left(1 - \frac{1}{e^2}\right),\end{aligned}$$



where we have used the substitution $x = 2t$, the fact that sin is symmetric about $\pi/2$, and the Jordan inequality. Now clearly this last expression goes to zero as $R \to \infty$, so

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz^2}}{z^8+1}\, dz = 0.$$

Since the only singularities of the integrand inside the curve $L_R + C_R + L'_R$ are simple poles at $z = e^{i\pi/8}$ and $z = e^{3i\pi/8}$, we have

$$\int_{L_R} \frac{e^{iz^2}}{z^8+1}\, dz + \int_{L'_R} \frac{e^{iz^2}}{z^8+1}\, dz + \int_{C_R} \frac{e^{iz^2}}{z^8+1}\, dz = 2\pi i \left[\mathrm{Res}_{e^{i\pi/8}} \frac{e^{iz^2}}{z^8+1} + \mathrm{Res}_{e^{3i\pi/8}} \frac{e^{iz^2}}{z^8+1} \right]. \tag{2}$$

These residues may be calculated as follows. If z_0 denotes either of the poles, then

$$\mathrm{Res}_{z_0} \frac{e^{iz^2}}{z^8+1} = \lim_{z\to z_0} (z-z_0) \frac{e^{iz^2}}{z^8+1} = \lim_{z\to z_0} e^{iz^2} \frac{z-z_0}{(z^8+1)-(z_0^8+1)} = e^{iz_0^2} \frac{1}{\frac{d}{dz} z^8 + 1\big|_{z=z_0}} = \frac{e^{iz_0^2}}{8z_0^7},$$

so the residues in (2) above are

$$\begin{aligned} \frac{e^{ie^{i\pi/4}}}{8e^{7\pi i/8}} &= \frac{e^{-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}}{8e^{7\pi i/8}} = \frac{e^{-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}}{8}(-i)e^{-3\pi i/8}, \\ \frac{e^{ie^{3\pi i/4}}}{8e^{21\pi i/8}} &= -\frac{e^{-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}}}{8} e^{3\pi i/8}; \end{aligned}$$

if we let $\omega = a + ib$ denote the second of these, then the sum of residues becomes

$$\omega + i\overline{\omega} = a + ib + i(a - ib) = a(1+i) + ib(1-i),$$

so the sum of integrals in (2) will become

$$\begin{aligned} 2\pi i(a(1+i) + ib(1-i)) &= 2\pi[-a(1-i) - b(1-i)] = -2\pi(a+b)(1-i) \\ &= \frac{2\pi}{8}(1-i)e^{-\frac{1}{\sqrt{2}}} \left[\cos\left(\frac{3\pi}{8} - \frac{1}{\sqrt{2}}\right) + \sin\left(\frac{3\pi}{8} - \frac{1}{\sqrt{2}}\right) \right]. \end{aligned} \tag{3}$$

We must now determine how to evaluate the integral over L'_R. If we parameterise $-L'_R$ as it, $t \in [0, R]$, then we may write

$$\int_{L'_R} \frac{e^{iz^2}}{z^8+1}\, dz = -i \int_0^R \frac{e^{-it^2}}{t^8+1}\, dt,$$

whence we see that

$$\int_{L_R} \frac{e^{iz^2}}{z^8+1}\,dz + \int_{L_R'} \frac{e^{iz^2}}{z^8+1}\,dz = \int_0^R \frac{e^{it^2} - ie^{-it^2}}{t^8+1}\,dt$$
$$= \int_0^R \frac{\cos t^2 - \sin t^2 - i(\cos t^2 - \sin t^2)}{t^8+1}\,dt = (1-i)\int_0^R \frac{\cos t^2 - \sin t^2}{t^8+1}\,dt,$$

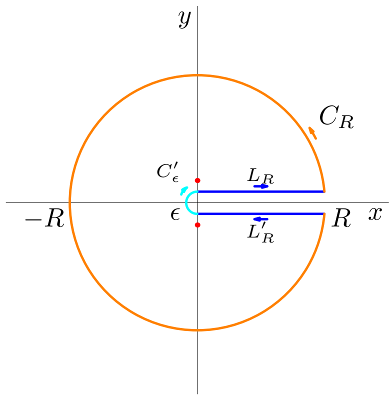
and we have finally from (3)

$$\int_{-\infty}^{\infty} \frac{\cos t^2 - \sin t^2}{t^8+1}\,dt = \frac{2\pi}{8} e^{-\frac{1}{\sqrt{2}}} \left[\cos\left(\frac{3\pi}{8} - \frac{1}{\sqrt{2}}\right) + \sin\left(\frac{3\pi}{8} - \frac{1}{\sqrt{2}}\right)\right].$$

[Marking: 2 marks for the choice of curves; 1 mark for applying the residue theorem; 3 marks for showing the integral over C_R goes to 0 as $R \to \infty$; 4 marks for the residue calculations; 2 marks for the calculation of the sum of the residues; 3 marks for evaluating L_R' and obtaining the final result.]

The second integral is rather easier. Since we wish as usual to evaluate the integral by using a contour in the complex plane, we must pick a particular branch of the square root function appearing in the numerator and choose a contour which avoids the corresponding branch cut. As in the example in the notes, we shall take a branch cut along the positive real axis, and require the angle for the square root function to lie in the interval $(0, 2\pi)$. Having done this, we shall use the keyhole contour shown in the figure. Now for R large, we may write on C_R, noting that $|z^{1/2}| = \sqrt{|z|}$ no matter which branch of the square root function we use,

$$R\left|\frac{z^{1/2}}{1+z^2}\right| \le R\frac{R^{1/2}}{R^2-1}, \tag{4}$$



which clearly goes to zero as $R \to \infty$; thus we must have

$$\lim_{R\to\infty} \int_{C_R} \frac{z^{1/2}}{1+z^2}\,dz = 0.$$

Similarly, for ϵ very small we have a result exactly analogous to (4):

$$\epsilon\left|\frac{z^{1/2}}{1+z^2}\right| \le \epsilon\frac{e^{1/2}}{1-\epsilon^2},$$

where we have $|1+z^2| \ge 1-\epsilon^2$ since we are interested in ϵ small and may assume $\epsilon < 1$. This goes to zero as $\epsilon \to 0$, which implies that

$$\lim_{\epsilon\to 0^+} \int_{C_\epsilon'} \frac{z^{1/2}}{1+z^2}\,dz = 0.$$

Thus in the limits $R \to \infty$, $\epsilon \to 0$, we have, since the only poles of the integrand inside the contour are at $\pm i$, and the integrand has no other singularities inside the contour,

$$\int_{L_1} \frac{z^{1/2}}{1+z^2}\,dz + \int_{L_2} \frac{z^{1/2}}{1+z^2}\,dz = 2\pi i\left[\mathrm{Res}_i \frac{z^{1/2}}{1+z^2} + \mathrm{Res}_{-i}\frac{z^{1/2}}{1+z^2}\right]. \tag{5}$$

Now

$$\begin{aligned}
\mathrm{Res}_i \frac{z^{1/2}}{1+z^2} &= \lim_{z\to i}(z-i)\frac{z^{1/2}}{(z-i)(z+i)} \\
&= \lim_{z\to i}\frac{z^{1/2}}{z+i} = \frac{e^{i\pi/4}}{2i} = \frac{1}{2\sqrt{2}} - i\frac{1}{2\sqrt{2}}, \\
\mathrm{Res}_{-i} \frac{z^{1/2}}{1+z^2} &= \lim_{z\to -i}(z+i)\frac{z^{1/2}}{(z-i)(z+i)} \\
&= \lim_{z\to -i}\frac{z^{1/2}}{z-i} = \frac{e^{3i\pi/4}}{-2i} = -\frac{1}{2\sqrt{2}} - i\frac{1}{2\sqrt{2}},
\end{aligned}$$

so the sum of residues in (5) is simply $-i/(2\sqrt{2})$ and the sum of integrals in (5) is $\pi/\sqrt{2}$. Now we may parameterise the lines L_1 and $-L_2$ by $t+i\epsilon$ and $t-i\epsilon$, respectively, where $t \in [0,R]$ in both cases. Now in polar form

$$\begin{aligned}
t+i\epsilon &= \sqrt{t^2+\epsilon^2}\,e^{i\arctan\frac{\epsilon}{t}}, \\
t-i\epsilon &= \sqrt{t^2+\epsilon^2}\,e^{i(2\pi-\arctan\frac{\epsilon}{t})},
\end{aligned}$$

where the range of arctan is $(-\pi/2, \pi/2)$ and the angle in the second line is chosen so as to lie in the interval $(0, 2\pi)$ corersponding to our chosen branch of $z^{1/2}$. Thus

$$\begin{aligned}
\int_{L_1} \frac{z^{1/2}}{1+z^2}\,dz &= \int_0^R \frac{\left(t^2+\epsilon^2\right)^{1/4} e^{\frac{1}{2}i\arctan\frac{\epsilon}{t}}}{1+(t+i\epsilon)^2}\,dt, \\
-\int_{L_2} \frac{z^{1/2}}{1+z^2}\,dz &= \int_0^R \frac{\left(t^2+\epsilon^2\right)^{1/4} e^{\frac{1}{2}i(2\pi-\arctan\frac{\epsilon}{t})}}{1+(t-i\epsilon)^2}\,dt;
\end{aligned}$$

taking the limit as $\epsilon \to 0^+$ gives

$$\begin{aligned}
\lim_{\epsilon\to 0^+}\int_{L_1} \frac{z^{1/2}}{1+z^2}\,dz &= \int_0^R \frac{\sqrt{t}}{1+t^2}\,dt, \\
-\lim_{\epsilon\to 0^+}\int_{L_2} \frac{z^{1/2}}{1+z^2}\,dz &= \int_0^R \frac{-\sqrt{t}}{1+t^2}\,dt,
\end{aligned}$$

so substituting in to equation (5) and taking the limit as $R \to \infty$ gives

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx = \frac{1}{2}\cdot\frac{\pi}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

[Marking: 1 mark for the contour; 2 marks each for showing the integrals over C_R and C'_ϵ vanish in the appropriate limits; 1 mark for applying the residue theorem; 2 marks for calculating the residues; 2 marks each for working out the L_R and L'_R integrals; 1 mark for adding up and solving.]