MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR JULY 13 – 17

Due Wednesday, July 22, at 3:30 PM EDT.

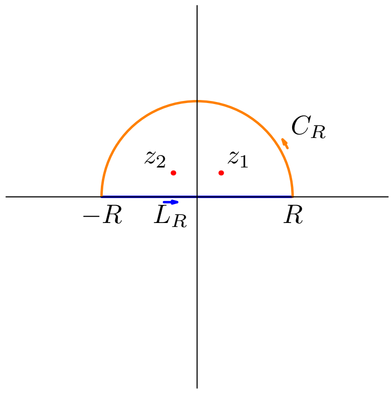
1. [20 marks] Evaluate the following integrals:

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4}\,dx, \qquad \int_{-\infty}^{+\infty} \frac{1}{1-x^2+x^4}\,dx.$$

(You may cite the term test solutions on the course website in your solution, if you wish.)

To evaluate the first integral, we proceed as follows. Let $f(z) = 1/(1+z^4)$, let $R > 1$, let L_R denote the line segment from $-R$ to R along the real axis, and let C_R denote the semicircle of radius R centred at 0 in the upper half-plane. Then the only poles of f inside the closed contour $L_R + C_R$ are at (see the picture)

$$z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \quad z_2 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}},$$



so that we may write, by the residue theorem,

$$\int_{L_R} \frac{1}{1+z^4}\,dz + \int_{C_R} \frac{1}{1+z^4}\,dz = 2\pi i \left[\mathrm{Res}_{z_1} \frac{1}{1+z^4} + \mathrm{Res}_{z_2} \frac{1}{1+z^4}\right].$$

Moreover, if $z = Re^{it}$, $t \in [0, \pi]$ is some point in C_R, then we have

$$R|f(z)| = R\left|\frac{1}{1+R^4e^{4it}}\right| \leq \frac{R}{R^4-1} = \frac{R^{-3}}{1-R^{-4}},$$

which clearly goes to zero as $R \to \infty$. Thus we have

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{1+z^4}\,dz = 0$$

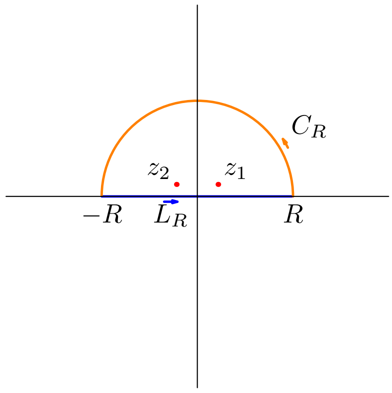
as well, and therefore our original integral is

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4}\,dx = \lim_{R\to\infty} \int_{L_R} \frac{1}{1+z^4}\,dz = 2\pi i \left[\mathrm{Res}_{z_1} \frac{1}{1+z^4} + \mathrm{Res}_{z_2} \frac{1}{1+z^4}\right] = \frac{\pi}{\sqrt{2}},$$

where the final answer follows from question 3(c) on the main sitting of the term test.

To evaluate the second integral, we proceed in an analogous fashion. Let $f(z) = 1/(1-z^2+z^4)$, let $R > 1$, let L_R denote the line segment from $-R$ to R along the real axis, and let C_R denote the semicircle of radius R centred at 0 in the upper half-plane. Then the only poles of f inside the closed contour $L_R + C_R$ are at (see the picture)

$$z_1 = \frac{\sqrt{3}}{2} + i\frac{1}{2}, \quad z_2 = -\frac{\sqrt{3}}{2} + i\frac{1}{2},$$



so that we may write, by the residue theorem,

$$\int_{L_R} \frac{1}{1-z^2+z^4}\,dz + \int_{C_R} \frac{1}{1-z^2+z^4}\,dz = 2\pi i \left[\mathrm{Res}_{z_1}\frac{1}{1-z^2+z^4} + \mathrm{Res}_{z_2}\frac{1}{1-z^2+z^4}\right].$$

Moreover, if $z = Re^{it}$, $t \in [0,\pi]$ is some point in C_R, then we have

$$R|f(z)| = R\left|\frac{1}{1-R^2e^{2it}+R^4e^{4it}}\right| \le \frac{R}{R^4-R^2-1} = \frac{R^{-3}}{1-R^{-2}-R^{-4}},$$

which clearly goes to zero as $R \to \infty$. Thus we have

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{1-z^2+z^4}\,dz = 0$$

as well, and therefore our original integral is

$$\int_{-\infty}^{+\infty} \frac{1}{1-x^2+x^4}\,dx = \lim_{R\to\infty} \int_{L_R} \frac{1}{1-z^2+z^4}\,dz = 2\pi i \left[\mathrm{Res}_{z_1}\frac{1}{1-z^2+z^4} + \mathrm{Res}_{z_2}\frac{1}{1-z^2+z^4}\right] = \pi$$

where the final answer follows from question 3(c) on the makeup sitting of the term test.

[Marking: for each integral, 2 marks for the setup (description or picture of L_R, C_R, and the poles); 2 marks for the residue theorem (or the Cauchy integral theorem, to relate it to the integral on the term test); 3 marks for showing that the integral over C_R goes to zero; 3 marks for deducing the final value.]

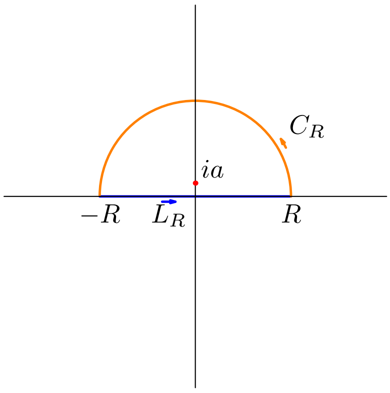
2. [30 marks] Evaluate the following integrals:

$$\int_{-\infty}^{+\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx, \qquad m,\ a \text{ real},\ a \neq 0.$$

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^4}\,dx, \qquad k \text{ any real number. [Hint: Apply your work from problem 1.]}$$

As mentioned in the announcement of July 21, for the first integral all that is required is to set things up and then evaluate the residue, ignoring the singularity at $z = 0$. We thus let L_R be the line segment from $-R$ to R, and C_R the semicircle centred at 0 with radius R in the upper half-plane; then we see that the only singularity the integrand has inside the closed curve $L_R + C_R$ is at $z = ia$ (where we assume that $a > 0$ without loss of generality). (See the picture.) Now by the residue theorem we have

$$\int_{L_R} \frac{\sin mz}{z(z^2+a^2)}\,dz + \int_{C_R} \frac{\sin mz}{z(z^2+a^2)}\,dz = 2\pi i \mathrm{Res}_{ia}\frac{\sin mz}{z(z^2+a^2)}.$$



We leave the issue of how to deal with the integral over C_R for later and simply show how to calculate the residue. Since $z^2 + a^2 = (z - ia)(z + ia)$, we see that $z = ia$ is a simple pole of $(\sin mz)/[z(z^2 + a^2)]$, and thus the residue may be calculated as follows:

$$\begin{aligned}\mathrm{Res}_{ia} \frac{\sin mz}{z(z^2+a^2)} &= \lim_{z \to ia} (z - ia) \frac{\sin mz}{z(z^2+a^2)} = \lim_{z \to ia} \frac{\sin mz}{z(z+ia)} \\ &= -\frac{\sin mia}{2a^2} = -\frac{i \sinh ma}{2a^2},\end{aligned}$$

whence we see that

$$2\pi i \mathrm{Res}_{ia} \frac{\sin mz}{z(z^2+a^2)} = \frac{\pi \sinh ma}{a^2}.$$

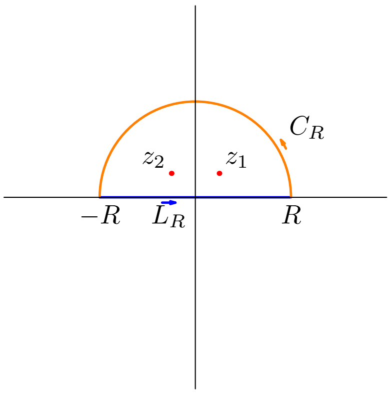
[Marking: again, 2 marks for the setup, 2 marks for the residue theorem; then 3 marks for computing the residue.]

Finally, for the last integral we must consider two separate cases depending on whether $k > 0$ or $k < 0$. (For $k = 0$ this is simply the integral from question 1.) Suppose that $k > 0$, and let L_R denote the line segment from $-R$ to R and C_R the semicircle centred at 0 with radius R in the *upper* half-plane, as usual; then for $R > 1$ the only poles will be those at

$$z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \quad z_2 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}},$$

as found in question 1 (see the picture), and by the residue theorem we have as usual

$$\int_{L_R} \frac{e^{ikz}}{1+z^4}\, dz + \int_{C_R} \frac{e^{ikz}}{1+z^4}\, dz = 2\pi i \left[\mathrm{Res}_{z_1} \frac{e^{ikz}}{1+z^4} + \mathrm{Res}_{z_2} \frac{e^{ikz}}{1+z^4}\right].$$



Now note that for $R > 1$ we have, for any $z = Re^{it}$ on C_R,

$$\left|\frac{1}{1+z^4}\right| = \left|\frac{1}{1+R^4 e^{4it}}\right| \le \frac{1}{R^4 - 1},$$

which clearly goes to zero as $R \to \infty$; thus by the Jordan lemma we must have $\int_{C_R} \frac{e^{ikz}}{1+z^4}\,dz \to 0$ as $R \to \infty$. Thus we are left with computing the residues. Now the poles are all of order 1, and thus we have

$$\mathrm{Res}_{z_i} \frac{e^{ikz}}{1+z^4} = \lim_{z \to z_i} (z - z_i) \frac{e^{ikz}}{1+z^4} = e^{ikz_i} \lim_{z \to z_i} (z - z_i) \frac{1}{1+z^4}, \tag{1}$$

where we have used the product rule for limits and the continuity of the exponential function. Now from the term test we have

$$\begin{aligned}
\mathrm{Res}_{z_1} \frac{1}{1+z^4} &= \lim_{z \to z_1} (z - z_1) \frac{1}{1+z^4} = \frac{1}{2\pi i} \cdot \frac{2\pi i}{2^{3/2}(-1+i)} = -\frac{1+i}{2^{5/2}}, \\
\mathrm{Res}_{z_2} \frac{1}{1+z^4} &= \lim_{z \to z_2} (z - z_2) \frac{1}{1+z^4} = \frac{1}{2\pi i} \cdot \frac{2\pi i}{2^{3/2}(1+i)} = \frac{1-i}{2^{5/2}};
\end{aligned}$$

thus we have finally

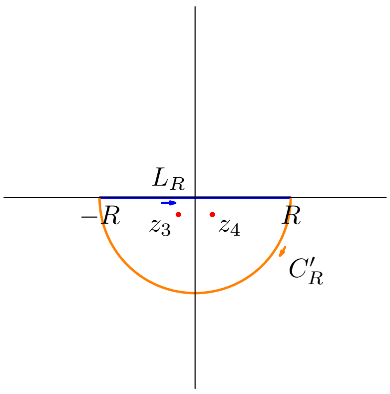
$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^4}\,dx &= 2\pi i \left[\mathrm{Res}_{z_1} \frac{e^{ikz}}{1+z^4} + \mathrm{Res}_{z_2} \frac{e^{ikz}}{1+z^4} \right] \\
&= 2\pi i \left[-e^{k\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)} \frac{1+i}{2^{5/2}} + e^{k\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)} \frac{1-i}{2^{5/2}} \right] \\
&= -4\pi \mathrm{Im}\, e^{-\frac{k}{\sqrt{2}} - i\frac{k}{\sqrt{2}}} \frac{1-i}{2^{5/2}} = \frac{\pi}{\sqrt{2}} e^{-k/\sqrt{2}} \left(\cos \frac{k}{\sqrt{2}} + \sin \frac{k}{\sqrt{2}} \right).
\end{aligned}$$

The integral for $k < 0$ is analogous except that now we close using the semicircle C_R' in the *lower* half-plane, as shown in the figure below; this means that we pick up the poles at

$$z_3 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}, \qquad z_4 = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}.$$

As before we have by the residue theorem that

$$\int_{L_R} \frac{e^{ikz}}{1+z^4}\,dz + \int_{C_R'} \frac{e^{ikz}}{1+z^4}\,dz = -2\pi i \left[\mathrm{Res}_{z_3} \frac{e^{ikz}}{1+z^4} + \mathrm{Res}_{z_4} \frac{e^{ikz}}{1+z^4} \right],$$



where the minus sign is required since the curve $L_R + C_R'$ is now oriented *clockwise*. Now by a straightforward modification of Jordan's lemma to the case where $k < 0$ and we close in the *lower* half-plane, since on C_R' we have ($z = Re^{it}$)

$$\left| \frac{1}{1+z^4} \right| = \left| \frac{1}{1+R^4 e^{4it}} \right| \leq \frac{1}{R^4 - 1}$$

as before, we have that $\int_{C'_R} \frac{e^{ikz}}{1+z^4}\,dz \to 0$ as $R \to \infty$. Now formula (1) holds just as well for $i = 3, 4$ as for $i = 1, 2$; since from the term test we have

$$\mathrm{Res}_{z_3} \frac{1}{1+z^4} = \lim_{z \to z_3} (z - z_3) \frac{1}{1+z^4} = \frac{1}{2\pi i} \cdot \frac{2\pi i}{2^{3/2}(1-i)} = \frac{1+i}{2^{5/2}},$$
$$\mathrm{Res}_{z_4} \frac{1}{1+z^4} = \lim_{z \to z_4} (z - z_4) \frac{1}{1+z^4} = \frac{1}{2\pi i} \cdot \frac{2\pi i}{2^{3/2}(-1-i)} = \frac{-1+i}{2^{5/2}},$$

we have finally in this case

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^4}\,dx &= -2\pi i \left[\mathrm{Res}_{z_3} \frac{e^{ikz}}{1+z^4} + \mathrm{Res}_{z_4} \frac{e^{ikz}}{1+z^4}\right] \\ &= -2\pi i \left[e^{k\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)} \frac{1+i}{2^{5/2}} - e^{k\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)} \frac{1-i}{2^{5/2}}\right] \\ &= 4\pi \mathrm{Im}\, e^{\frac{k}{\sqrt{2}} - i\frac{k}{\sqrt{2}}} \frac{1+i}{2^{5/2}} = \frac{\pi}{\sqrt{2}} e^{k/\sqrt{2}} \left(\cos\frac{k}{\sqrt{2}} - \sin\frac{k}{\sqrt{2}}\right).\end{aligned}$$

We see that we may combine these two expressions into one as follows:

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}} e^{-|k|/\sqrt{2}} \left(\cos\frac{k}{\sqrt{2}} + \sin\frac{|k|}{\sqrt{2}}\right).$$

[Marking: 11 marks for each, plus 1 mark for getting the extra minus sign in the second integral. For each integral, 1 mark for the setup, 1 mark for the residue theorem, 1 mark for the application of Jordan's lemma, 2 marks for each residue, and 4 marks for the final computations.]