MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR JULY 6 – 10

Due Tuesday, July 14, at 3:30 PM EDT.

1. Using the formulas in Goursat, §§35, 37, find the Laurent series for

$$f(z) = \frac{\sin z}{z}$$

around $z = 0$. How does this series compare to the Taylor series for $\sin z$ around $z = 0$? On what set does it converge? Justify your answer.

Let us write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

and let γ be a circle around $z = 0$; then we have the formulas

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z')}{(z'-0)^{n+1}}\, dz'$$
$$b_n = \frac{1}{2\pi i} \int_\gamma f(z')(z'-0)^{n-1}\, dz'.$$

Substituting in $f(z) = (\sin z)/z$, the first of these gives

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{\sin z'}{z'^{\,n+2}}\, dz' \text{[1 mark]}$$
$$= \frac{1}{2\pi i} \cdot \frac{2\pi i}{(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \sin z\Big|_{z=0} = \frac{1}{(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \sin z\Big|_{z=0}; \text{[2 marks]}$$

thus $a_0 = 1$, $a_1 = 0$, $a_2 = -1/6$, $a_3 = 0$, etc., and in general $a_{2k+1} = 0$ while $a_{2k} = (-1)^k/(2k+1)!$[1 mark]. Now

$$b_1 = \frac{1}{2\pi i} \int_\gamma \frac{\sin z'}{z'}\, dz' \text{[1 mark]} = \frac{1}{2\pi i} \sin 0 = 0, \text{[1 mark]}$$

while if $n > 1$

$$b_n = \frac{1}{2\pi i} \int_\gamma \sin z' (z'-0)^{n-2}\, dz' = 0 \text{[1 mark]}$$

by the Cauchy integral theorem, since the integrand is analytic everywhere. Thus we have

$$\frac{\sin z}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}. \text{[1 mark]}$$

Now recall that the Taylor series for $\sin z$ is

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1};$$

in other words, the Laurent series for $(\sin z)/z$ is just the Taylor series for $\sin z$, divided term-by-term by z, as we might expect[1 mark]. Since the Taylor series for $\sin z$ converges everywhere on the complex plane, the Laurent series above will converge to an analytic function everywhere except possibly at $z = 0$; but at $z = 0$ the series also clearly converges to 1, so the series converges on the entire complex plane.[1 mark]

2. Again using the formulas in Goursat, §§35, 37, find the Laurent series for

$$f(z) = \frac{e^z}{(z-2)^2}$$

around $z = 2$. How does this compare to the Taylor series for e^z around $z = 2$? [Hint: recall that $e^{a+b} = e^a e^b$; can you use this to find the Taylor series?] On what set does this Laurent series converge? Again, justify your answer.

We use the same formulas as in question 1. Thus we have, first of all, letting now γ denote a circle around $z = 2$,

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{(z'-2)^{n+1}}\,dz' \\ &= \frac{1}{2\pi i}\int_\gamma \frac{e^{z'}}{(z'-2)^{n+3}}\,dz' \text{[1 mark]} = \frac{1}{(n+3)!}\,\frac{d^{n+2}}{dz^{n+2}}e^z\bigg|_{z=2} \text{[2 marks]} = \frac{e^2}{(n+2)!}, \text{[1 mark]} \end{aligned}$$

by the Cauchy integral formula for derivatives, while

$$\begin{aligned} b_1 &= \frac{1}{2\pi i}\int_\gamma f(z')\,dz' \text{[1 mark]} = \frac{1}{2\pi i}\int_\gamma \frac{e^{z'}}{(z'-2)^2}\,dz' = e^2, \text{[1 mark]} \\ b_2 &= \frac{1}{2\pi i}\int_\gamma f(z')(z'-2)\,dz' = \frac{1}{2\pi i}\int_\gamma \frac{e^{z'}}{z'-2}\,dz' = e^2, \text{[1 mark]} \end{aligned}$$

while for $n > 2$

$$b_n = \frac{1}{2\pi i}\int_\gamma f(z')(z'-2)^{n-1}\,dz' = \frac{1}{2\pi i}\int_\gamma e^{z'}(z'-2)^{n-3}\,dz' = 0 \text{[1 mark]}$$

by the Cauchy integral theorem. Thus we have the expansion

$$f(z) = \frac{e^2}{(z-2)^2} + \frac{e^2}{z-2} + \sum_{n=0}^{\infty} \frac{e^2}{(n+2)!}(z-2)^n. \text{[2 marks]}$$

Now the Taylor series for e^z around $z = 2$ can be found as follows:

$$e^z = e^2 e^{z-2} = e^2 \sum_{n=0}^{\infty} \frac{1}{n!}(z-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!}(z-2)^n; \text{[2 marks]}$$

from this we see that the above series is simply this series, divided by $(z-2)^2$ term-by-term, as we would expect.[1 mark]

Now the Taylor series for e^z around $z = 2$ converges on the entire complex plane; thus the series above for f will converge everywhere where it is defined, i.e., on the punctured plane $\{z \in \mathbf{C} \mid z \neq 2\}$.[2 marks]