MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR JUNE 8 – 12

Due Tuesday, June 16, at 3:30 PM EDT.

1. [9 marks] Let

$$f(x+iy) = \cos x \cosh y - i \sin x \sinh y.$$

By choosing and parameterising an appropriate (potentially only piecewise-smooth) curve, determine the function

$$F(z) = \int_0^z f(z')\, dz'.$$

Explain why the result does not depend on your choice of curve.

While there are many different choices of curve we could make, it is probably simplest to choose a piecewise-smooth curve each part of which is parallel to one of the coordinate axes, since in this way on each portion of the curve only one of the coordinates x and y will change at a time and we will only have to integrate single trigonometric or hyperbolic trigonometric functions. Specifically, then, let us suppose that the complex number z can be written as $z = x + iy$ and define our curve γ to be

$$\gamma(t) = \begin{cases} t, & t \in [0, x] \\ x + it, & t \in [0, y] \end{cases};$$

thus γ gives first a line along the real axis from 0 to the real part of z and then a line parallel to the imaginary axis from there to z. Since the integral along a piecewise-smooth curve is equal to the sum of the integrals along the different pieces, we may then write

$$\begin{aligned}
\int_\gamma f(z')\, dz' &= \int_0^x f(t) \frac{d}{dt} t\, dt + \int_0^y f(x+it) \frac{d}{dt}(x+it)\, dt \\
&= \int_0^x \cos t \cosh 0 - i \sin t \sinh 0\, dt + \int_0^y [\cos x \cosh t - i \sin x \sinh t]\,(i)\, dt \\
&= [\sin t]|_0^x + i\,[\cos x \sinh t - i \sin x \cosh t]|_0^y \\
&= \sin x + i\,[\cos x \sinh y - i \sin x \cosh y - (-i \sin x)] \\
&= \sin x - \sin x + \sin x \cosh y + i \cos x \sinh y \\
&= \sin x \cosh y + i \cos x \sinh y.
\end{aligned}$$

This integral will be independent of the path chosen by the Cauchy integral theorem, since the function $f(z)$ is analytic everywhere in the plane – this follows from the Cauchy-Riemann equations, or from noting that $f(z) = \cos z$.

Having noted that $f(z) = \cos z$, we note also that $F(z) = \sin z$, so that F is indeed an antiderivative of f, as it should be.

Marking: 2 marks for choosing a piecewise-smooth path from 0 to z; 5 marks for evaluating the integral (roughly as follows: 1 mark each for calculating γ' and substituting in γ correctly, 3 marks for actually evaluating the t integrals); 1 mark for invoking the Cauchy integral theorem (or similar argument), 1 mark for noting that this holds since f is analytic (some justification for f (some justification is required).

2. [9 marks] Using the Cauchy integral formula, evaluate the following integrals:

$$\int_\gamma \frac{\cos z}{z}\, dz,$$ γ the square with sidelength 2 centred at the origin, oriented counterclockwise.

$$\int_\gamma \frac{1}{(z-z_0)^2}\, dz,$$ γ any simple closed curve containing the point z_0, in any orientation.

$$\int_\gamma \frac{e^z}{z}\, dz,$$ γ the unit circle, oriented *clockwise*.

We recall the Cauchy integral formula. If f is a function which is analytic on and within a simple closed curve C, and z_0 is some point within this curve, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'.$$

This formula applies to smooth and piecewise-smooth curves.

Thus for the first integral we may take $n = 0$, $C = \gamma$, $z_0 = 0$, and $f(z) = \cos z$ to obtain

$$\int_\gamma \frac{\cos z}{z}\,dz = \int_\gamma \frac{f(z')}{z'-z_0}\,dz' = 2\pi i f(z_0) = 2\pi i.$$

For the second integral, we may take $n = 1$, $C = \gamma$, $z_0 = z_0$, and $f(z) = 1$ to obtain

$$\int_\gamma \frac{1}{(z-z_0)^2}\,dz = \int_\gamma \frac{f(z')}{(z'-z_0)^2}\,dz' = \frac{2\pi i}{1!}f'(z_0) = 0,$$

since $f'(z) = 0$ for all z as f is constant.

Finally, for the third integral we may take $n = 0$, $z_0 = 0$, $f(z) = e^z$; taking C to be γ in the *opposite* orientation, hence oriented counterclockwise, and noting that changing the notation only changes the integral by introducing an extra minus sign, we have

$$\int_\gamma \frac{e^z}{z}\,dz = -\int_C \frac{f(z')}{z'-z_0}\,dz' = -2\pi i f(z_0) = -2\pi i e^0 = -2\pi i.$$

Marking: For the first integral, 1 mark for identifying f, 1 mark for identifying $z_0 = 0$, 1 mark for the final answer. For the second integral, 1 mark for identifying $n = 1$, 1 mark for identifying f, 1 mark for the final answer. For the third integral, 1 mark for identifying f, 1 mark for the minus sign, 1 mark for the final answer.

3. [6 marks] Let γ denote the unit circle, oriented counterclockwise, and let $z^{1/2}$ denote any branch of the square root function (be sure to clearly indicate which one you are using!). By direct computation, evaluate the integral

$$\int_\gamma \frac{z^{1/2}}{z}\,dz,$$

where we can evaluate the integral since the function is defined and bounded everywhere except at a single point on the curve (alternatively, you can view the above integral as a limit of an open segment of the circle as the two endpoints come towards the branch cut). Does your result contradict the Cauchy integral theorem or formula? Why or why not?

Just to make things interesting, let us take the branch obtained by cutting along the line $\theta = 5\pi/4$ and requiring the angle to lie in $(-3\pi/4, 5\pi/4)$. Note that we may parameterise the unit circle by

$$\gamma(t) = \cos t + i\sin t, \quad t \in [\theta_0, \theta_0 + 2\pi],$$

where θ_0 is any real number. Since by our choice of branch the angle – and hence the parameter value t – must lie in the interval $(-3\pi/4, 5\pi/4)$, we choose $\theta_0 = -3\pi/4$. We still have to figure out how to perform the integral when the function is not defined on the branch cut itself. There are a couple ways of looking at this. Probably the most rigorous one is that given in the parenthesis in the problem statement: essentially, consider the resulting t integral as an improper integral and evaluate it by taking limits towards the endpoints. Since along γ we may write

$$z^{1/2} = \cos\frac{t}{2} + i\sin\frac{t}{2},$$

this gives

$$\lim_{L_1 \to -3\pi/4^-} \lim_{L_2 \to 5\pi/4^+} \int_{L_1}^{L_2} \frac{\cos\frac{t}{2} + i\sin\frac{t}{2}}{\cos t + i\sin t} (-\sin t + i\cos t)\, dt;$$

i.e., instead of integrating from $-3\pi/4$ to $5\pi/4$, we integrate from some value L_1 to some other value L_2, both of which are inside the interval $(-3\pi/4, 5\pi/4)$, and then take the limit as they approach the two endpoints. The integral above may be evaluated as follows. Note that $\cos t + i\sin t = e^{it}$, so $1/(\cos t + i\sin t) = e^{-it} = \cos t - i\sin t$ (this can also be determined directly, using division of complex numbers, of course); thus the above integral equals

$$\begin{aligned}\int_{L_1}^{L_2} \left[\cos\frac{t}{2} + i\sin\frac{t}{2}\right](\cos t - i\sin t)(-\sin t + i\cos t)\, dt &= \int_{L_1}^{L_2} \left[\cos\frac{t}{2} + i\sin\frac{t}{2}\right] i\, dt \\ &= 2i\left[\sin\frac{t}{2} - i\cos\frac{t}{2}\right]\Big|_{L_1}^{L_2}.\end{aligned}$$

Now note that the result here is a continuous function of L_1 and L_2, so we may evaluate the limits above by substituting in the limiting values $L_1 = -3\pi/4$ and $L_2 = 5\pi/4$ (this is basically what the remark in the problem statement that 'we can evaluate the integral since the function is defined and bounded everywhere except at a single point on the curve' was getting at!); recalling that $\sin(x - \pi) = -\sin x$ and $\cos(x - \pi) = -\cos x$, we obtain

$$\begin{aligned}\int_\gamma \frac{z^{1/2}}{z}\, dz &= 2i\left[\left(\sin\frac{5\pi}{8} - i\cos\frac{5\pi}{8}\right) - \left(\sin -\frac{3\pi}{8} - i\cos -\frac{3\pi}{8}\right)\right] \\ &= 2i\left[2\sin\frac{5\pi}{8} - 2i\cos\frac{5\pi}{8}\right] = 4(\cos 5\pi/8 + i\sin 5\pi/8).\end{aligned}$$

Had we instead made a branch cut at $\theta = \alpha$, and required our angle to lie in $(\alpha - 2\pi, \alpha)$, we would evidently have obtained $4(\cos\alpha/2 + i\sin\alpha/2)$ instead. Note though that this number will never be 0 since it always lies on the circle of radius 4 centred at the origin. This does not contradict the Cauchy integral formula – which, if applied naively to the current integral, would have given $2\pi i 0^{1/2} = 0$ – since the function $z^{1/2}$ is not analytic at the origin, which lies inside the curve γ. (It is interesting to note that, had we integrated along a closed curve which wrapped around the origin *twice*, and used the full root function rather than a branch, the integral would have been zero. This is related to the fact that $z^{1/2}$ is double-valued. In this context, we should note that a closed curve which wraps twice around the origin cannot be continuously deformed into a curve wrapping once around the origin without passing through the origin.)

Marking: picking a branch; parameterising the curve with the correct interval; correct integrand; integrating; final answer; explanation, 1 mark each. Explicitly evaluating the integral as an improper one – as done above – was not required for full marks.