MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR JUNE 1 – 5

Due Tuesday, June 9, at 3:30 PM EDT.

1. [6 marks] Without doing any differentiation, explain why the following functions are harmonic on the indicated regions:

$$\begin{aligned} &\cos\left(\cos x \cosh y\right) \cosh\left(\sin x \sinh y\right), && \text{everywhere on the plane.} \\ &\frac{1}{2}\log\left(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y\right) && \text{on the set } \{(x,y)|x,y>0\}. \end{aligned}$$

We recall the following functions, which are analytic on the given regions:

$$\begin{aligned} \cos(x+iy) &= \cos x \cosh y - i \sin x \sinh y, && \text{everywhere on the plane} \\ \sin(x+iy) &= \sin x \cosh y + i \cos x \sinh y, && \text{everywhere on the plane} \end{aligned}$$

moreover, the complex logarithm

$$\operatorname{Log}(x+iy) = \frac{1}{2}\log\left(x^2+y^2\right) + i\theta(x,y)$$

will be analytic on any cut plane, where $\theta(x,y)$ denotes the angle for $x+iy$ corresponding to the particular choice of branch. Thus

$$\begin{aligned} \cos[\cos(x+iy)] &= \cos(\cos x \cosh y)\cosh(-\sin x \sinh y) - i\sin(\cos x \cosh y)\sinh(-\sin x \sinh y) \\ &= \cos(\cos x \cosh y)\cosh(\sin x \sinh y) + i\sin(\cos x \cosh y)\sinh(\sin x \sinh y) \end{aligned}$$

will be analytic everywhere on the plane, and its real part

$$\cos(\cos x \cosh y)\cosh(\sin x \sinh y)$$

will thus be harmonic everywhere on the plane, as required. Further, for every point $x+iy \neq 0$

$$\operatorname{Log}[\sin(x+iy)] = \frac{1}{2}\log\left(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y\right) + i\theta(\cos x \sinh y, \sin x \cosh y)$$

will be analytic in some disk around $x+iy$. Here we must take an appropriate branch to ensure single-valuedness; but since the choice of branch only affects the imaginary part, not the real part, we see that the real part will be harmonic at every point where $\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \neq 0$, and this is satisfied on $\{(x,y)|x,y>0\}$, since $\sinh y, \cosh y > 0$ on that region and $\sin x, \cos x$ are never simultaneously zero.

Marking: First function: 1 mark for identifying cos, 1 mark for noting that it is $\cos(\cos(x+iy))$ and hence has a harmonic real part. Second function: 1 mark for recognising Log, 1 mark for recognising sin, 1 mark for talking about the branch, 1 mark for concluding that the real part is therefore harmonic.

2. [18 marks] Evaluate the following integrals:

$$\int_\gamma \frac{1}{z}\,dz, \qquad \text{where } \gamma \text{ represents the unit circle, traversed counterclockwise.}$$

$$\int_\gamma \frac{1}{z}\,dz, \qquad \text{where } \gamma \text{ represents the circle of radius one and centre } 2i\text{, traversed counterclockwise.}$$

$$\int_\gamma \frac{1}{z^2}\,dz, \qquad \text{where } \gamma \text{ represents any circle centred at the origin.}$$

Do any of these results contradict the result we derived in class about integrals of analytic functions over closed curves? Do any of them add to that result? Why or why not?

The unit circle may be parameterised as

$$z(t) = \cos t + i \sin t = e^{it}, \text{[1 mark]} \qquad t \in [0, 2\pi]\text{[1 mark]};$$

the integral may be evaluated as follows:

$$\int_\gamma \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{z(t)} z'(t)\, dt\text{[1 mark]} = \int_0^{2\pi} e^{-it} \cdot ie^{it}\, dt\text{[1 marks]} = \int_0^{2\pi} i\, dt = 2\pi i.\text{[1 mark]}$$

Alternatively, we may write

$$\begin{aligned}
\int_\gamma \frac{1}{z}\, dz &= \int_0^{2\pi} \frac{1}{\cos t + i \sin t}(-\sin t + i \cos t)\, dt = \int_0^{2\pi} \frac{\cos t - i \sin t}{\cos^2 t + \sin^2 t}(-\sin t + i \cos t)\, dt \\
&= \int_0^{2\pi} \frac{-\cos t \sin t + \sin t \cos t + i(\cos^2 t + \sin^2 t)}{\cos^2 t + \sin^2 t}\, dt \\
&= \int_0^{2\pi} i\, dt = 2\pi i.
\end{aligned}$$

These two methods are entirely equivalent, though obviously the first is much shorter!

For the second integral, we note that that the circle of radius one and centre $2i$ does not include the origin[1 mark], which means that the function $\frac{1}{z}$ is analytic everywhere on the interior of the curve γ[2 marks], so by the Cauchy integral theorem this integral is zero.[2 marks]

For the final integral, let r denote the radius of the circle, Then we may parameterise γ as

$$z(t) = re^{it}, \text{[1 mark]} \qquad t \in [0, 2\pi], \text{[1 mark]}$$

and the integral is

$$\int_\gamma \frac{1}{z^2}\, dz = \int_0^{2\pi} \frac{1}{r^2 e^{2it}} rie^{it}\, dt\text{[1 mark]} = \int_0^{2\pi} \frac{1}{r} ie^{-it}\, dt\text{[1 marks]} = -\frac{1}{r}e^{-it}\Big|_0^{2\pi} = 0.\text{[1 mark]}$$

The Cauchy integral theorem states that if a function is analytic everywhere inside a particular simple closed curve, then its integral over that curve must be zero. Thus the second and third integrals do not contradict this theorem since both integrals are zero [1 mark]. The first integral does not contradict it either since the function $\frac{1}{z}$ is not analytic at $z = 0$, which lies inside the curve. [1 mark]The third integral does add something to the Cauchy integral theorem, though, since the integral is zero even though the function is not analytic everywhere inside the curve.[1 mark]

Marking: 5 marks for the first and third integrals, as indicated. The marks are for the following points: (1) a correct parameterisation; (2) correct interval; (3) correctly substituting into the integral; (4) algebra; (5) correct final result. 5 marks for the third integral; for an integral solution, the scheme is the same as for the first and third integrals, while for the above solution marks are as indicated. Remaining three marks for final paragraph as indicated.