MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR MAY 25 – 29

SOLUTIONS

Due Tuesday, June 2, at 12:00 noon EDT.

1. [8 marks] Determine $\operatorname{Log} z$ for each of the following points and branch cuts:

(a) $z = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$, branch cut along $\theta = \pi$, interval $(-\pi, \pi)$.
We have $z = e^{i\frac{7\pi}{4}} = e^{-i\frac{\pi}{4}}$, so $\operatorname{Log} z = \log 1 - i\frac{\pi}{4} = -i\frac{\pi}{4}$.

(b) $z = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$, branch cut along $\theta = 0$, interval $(0, 2\pi)$.
We have $z = e^{i\frac{7\pi}{4}}$, so $\operatorname{Log} z = \log 1 + i\frac{7\pi}{4} = i\frac{7\pi}{4}$.

(c) $z = e$, branch cut along $\theta = \pi/2$, interval $(\pi/2, 5\pi/2)$.
We have $z = e \cdot e^0 = e \cdot e^{2\pi i}$, so $\operatorname{Log} z = \log e + 2\pi i = 1 + 2\pi i$.

(d) $z = e$, branch cut along $\theta = \pi$, interval $(-\pi, \pi)$.
We have $z = e \cdot e^0$, so $\operatorname{Log} z = \log e + i \cdot 0 = 1$.

Marking: for each part, 1 mark each for the correct real and imaginary part of $\operatorname{Log} z$.

2. [5 marks] Compute the following difference of limits:

$$\lim_{\theta \to 0^+} \operatorname{Log} re^{i\theta} - \lim_{\theta \to 2\pi^-} \operatorname{Log} re^{i\theta},$$

where $r > 0$. Does this difference depend on which branch of Log is used? What would happen if we considered instead the difference

$$\lim_{\theta \to \theta_0^+} \operatorname{Log} re^{i\theta} - \lim_{\theta \to (\theta_0 + 2\pi)^-} \operatorname{Log} re^{i\theta}$$

where θ_0 is any real number?

Strictly speaking, since

$$\operatorname{Log} re^{i\theta} = \log r + i(\theta + 2n\pi), \qquad n \in \mathbf{Z},$$

the first difference above would become a difference of *sets* of values, namely

$$\{\log r + i \cdot 2n\pi | n \in \mathbf{Z}\} - \{\log r + i \cdot (2m+1)\pi | m \in \mathbf{Z}\},$$

which gives simply $\{2ni\pi | n \in \mathbf{Z}\}$. What was intended, though, was that we consider a *particular branch* of the logarithm before taking the limit. Now recall that choosing a branch also involves choosing a particular interval for θ; now we wish this interval to have numbers close to but slightly above 0 as well as numbers close to but slightly below 2π, and since it must be of length 2π, it must be simply the interval $(0, 2\pi)$. If we use this interval, we obtain

$$\lim_{\theta \to 0^+} \operatorname{Log} re^{i\theta} - \lim_{\theta \to 2\pi^-} \operatorname{Log} re^{i\theta} = \log r - [\log r + 2\pi i] = -2\pi i.$$

Taken in this sense, there is only one branch for which the limit makes sense, so the second part of the question does not even make any sense. If, however, we consider the limits on θ not strictly as limits on θ but rather as limits on *points*, while still requiring that the full 2π range between the points be included in the θ interval, then we can take other branches, but only those for which the θ interval is $(2n\pi, 2(n+1)\pi)$ for some integer n. For such a branch, the difference in the limits is now (rewriting the limits as noted above to correspond to points)

$$\lim_{\theta \to 2n\pi^+} \operatorname{Log} re^{i\theta} - \lim_{\theta \to 2(n+1)\pi^-} \operatorname{Log} re^{i\theta} = \log r + 2n\pi i - [\log r + 2(n+1)\pi i] = -2\pi i,$$

so that the difference does not depend on the branch chosen.

The second part is very similar. If we consider the full Log function, it is quite easy to see that we will get the same set as the difference:

$$\lim_{\theta \to \theta_0^+} \mathrm{Log}\, re^{i\theta} - \lim_{\theta \to (\theta_0 + 2\pi)^-} \mathrm{Log}\, re^{i\theta} = \{\log r + i(\theta_0 + 2n\pi) | n \in \mathbf{Z}\} - \{\log r + i(\theta_0 + 2(m+1)\pi) | m \in \mathbf{Z}\}$$
$$= \{2n\pi i | n \in \mathbf{Z}\}.$$

If we consider a branch cut along $\theta = \theta_0$, and again consider the indicated limits as indicating limits on *points* rather than limits on θ, then we see that we must take the θ range to be of the form $(\theta_0 + 2n\pi, \theta_0 + 2(n+1)\pi)$, and the difference in limits will be

$$\lim_{\theta \to (\theta_0 + 2n\pi)^+} \mathrm{Log}\, re^{i\theta} - \lim_{\theta \to (\theta_0 + 2(n+1)\pi)^-} \mathrm{Log}\, re^{i\theta} = \log r + i(\theta_0 + 2n\pi) - [\log r + i(\theta_0 + 2(n+1)\pi)] = -2\pi i,$$

exactly as before. This difference also does not depend on which branch we take.

Marking: Roughly, 2 marks for each difference, 1 mark for making a correct statement about the dependence on the branch.

3. [4 marks] For a given complex number z, use the quadratic formula and the relation

$$\cos w = \frac{e^{iw} + e^{-iw}}{2}$$

to compute all complex numbers w satisfying $\cos w = z$.

Let us write $z = \cos w$; then we may proceed as follows:

$$z = \frac{e^{iw} + e^{-iw}}{2}$$
$$e^{iw} + e^{-iw} = 2z$$
$$e^{2iw} - 2ze^{iw} + 1 = 0 \qquad \text{[1 mark]}$$
$$e^{iw} = \frac{1}{2}\left(2z + (4z^2 - 4)^{1/2}\right) = z + (z^2 - 1)^{1/2} \qquad \text{[1 mark]}$$
$$iw = \mathrm{Log}\left(z + (z^2 - 1)^{1/2}\right) \qquad \text{[1 mark]}$$
$$w = \frac{1}{i}\mathrm{Log}\left(z + (z^2 - 1)^{1/2}\right). \qquad \text{[1 mark]}$$

4. We know that the exponential function e^z is analytic on the entire complex plane, and hence conformal at each point. Let us see what this map looks like in practice.

(a) [6 marks] Consider straight lines parallel to the real and imaginary axes. What is the image of these lines under the map $z \mapsto e^z$? (For example, if you parameterise the two lines as $\gamma_k(t)$, what kind of curve is $e^{\gamma_k(t)}$?) Sketch a couple representative examples (both the original lines and the image curves).

A straight line parallel to the real axis can be parameterised as

$$\gamma(t) = t + iy$$

for some real number y. Under the map $z \mapsto e^z$, this becomes

$$t \mapsto e^t e^{iy} = e^t(\cos y + i \sin y),$$

which is a ray from the origin (but not including the origin) going to infinity along the direction given by $\cos y + i \sin y$. Similarly, a straight line parallel to the imaginary axis can be parameterised as

$$\gamma(t) = x + it,$$

where x is some real number; under the map $z \mapsto e^z$, this becomes

$$t \mapsto e^x e^{it} = e^x(\cos t + i \sin t),$$

which is a circle centred at the origin with radius e^x.

Marking: For both lines, 2 marks for a full and correct description of the image curve. 1 mark for each of the corresponding sketches.

(b) [4 marks] Now consider two lines passing through the origin, making angles θ_1 and θ_2 with the positive real axis. What is the image of these two curves under the map $z \mapsto e^z$? Sketch the image curves for two particular values of θ_1 and θ_2 (neither of which is a multiple of $\pi/2$!).

A curve passing through the origin making an angle θ with the positive real axis can be parameterised as

$$\gamma(t) = t(\cos\theta + i \sin\theta);$$

one way of seeing this is to note that the complex number $\cos\theta + i\sin\theta$ corresponds to a unit vector which makes an angle θ with the positive real axis. Thus the image of the two given curves under the exponential map $z \mapsto e^z$ is

$$e^{t(\cos\theta_1 + i\sin\theta_1)} = e^{t\cos\theta_1}\left(\cos\left[t\sin\theta_1\right] + i\sin\left[t\sin\theta_1\right]\right)$$

and

$$e^{t(\cos\theta_2 + i\sin\theta_2)} = e^{t\cos\theta_2}\left(\cos\left[t\sin\theta_2\right] + i\sin\left[t\sin\theta_2\right]\right)$$

If there weren't the factors of $e^{t\cos\theta_1}$ and $e^{t\cos\theta_2}$, these would be circles centred at the origin; but this leading factor means that we get instead *spirals* – at least as long as $\cos\theta_k \neq 0$!

Marking: 2 marks (total) for determining the image curves; 1 mark for each of the graphs.

(c) [3 marks] How does your work from (a) and (b) exemplify the conformality of e^z?

For part (a), it is clear that rays from the origin and circles centred at the origin intersect at 90° angles, just as do lines parallel to the real and imaginary axes. For part (b), conformality would mean that the angle between the original lines at the origin is equal to the angle between the spirals *at the origin* – if they intersect anywhere else it doesn't matter. The angle between the spirals at the origin can be found by computing their tangent vectors. In complex form, these are

$$\left.\frac{d}{dt}e^{t(\cos\theta_1 + i\sin\theta_1)}\right|_{t=0} = \cos\theta_1 + i\sin\theta_1,$$
$$\left.\frac{d}{dt}e^{t(\cos\theta_2 + i\sin\theta_2)}\right|_{t=0} = \cos\theta_2 + i\sin\theta_2,$$

which are just the direction vectors for the original lines. Thus the angle between the spirals at the origin is the same as the angle between the original lines.

Marking: 2 marks for noting that the rays and circles in (a) intersect at 90° as do the original lines. 1 mark for observing that the spirals in (b) make the same angle at the origin as do the original lines.

5. We know that branches of root functions are analytic on their domains, and hence conformal there. Let us see how this works out in practice.

(a) [8 marks] Choose a particular branch of the square root function $z \mapsto z^{1/2}$. (Make sure you indicate your choice clearly!) Consider straight rays from the origin and circles centred on the origin; what is their image under this map? Derive formulas and sketch a couple representative examples (sketch both original and image curves). How does this exemplify the conformality of your particular branch of $z \mapsto z^{1/2}$?

We choose the branch obtained by making a cut along the negative real axis and requiring θ to lie in $(-\pi, \pi)$; thus we have

$$z^{1/2} = r^{1/2}e^{i\theta/2} \qquad \text{when } z = re^{i\theta},\ \theta \in (-\pi, \pi).$$

Now a straight ray from the origin which makes an angle of θ with the positive real axis can be parameterised as

$$\gamma(t) = t(\cos\theta + i\sin\theta), \qquad t \in (0, +\infty);$$

we may assume that $\theta \in (-\pi, \pi)$, which means that the image of this ray under the chosen branch of the square root function is simply

$$t \mapsto t^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right), \qquad t \in (0, +\infty)$$

which is still a ray from the origin, but at an angle of $\theta/2$ to the positive real axis instead of θ. Similarly, a circle of radius r centred on the origin can be parameterised as

$$\gamma(t) = r(\cos t + i\sin t), \qquad t \in [-\pi, \pi];$$

in order to find its image we drop the point $-r$, which corresponds to $t = \pm\pi$, and thence obtain the image curve

$$t \mapsto r^{1/2}\left(\cos\frac{t}{2} + i\sin\frac{t}{2}\right), \qquad t \in (-\pi, \pi),$$

which is a *semicircle* extending from $-\pi/2$ to $\pi/2$ and with radius $r^{1/2}$, though still centred at the origin.

Since straight rays from the origin intersect both circles and semicircles at $90°$ angles, we see that the angles between these curves is preserved by the square root function, as expected.

Marking: 2 marks for specifying the branch (both the location of the cut and the θ interval). 2 marks each for deriving an analytic representation of the image curves. 1 mark for both sketches, 1 mark for noting the conformality relation.

(b) [4 marks] Consider two rays from the origin which make an angle of less than π with each other. What is the angle between the images of these lines under the map from (a)? Does this contradict what we know about the relationship between analytic functions and conformal maps? Why or why not?

Note that any two rays from the origin make two angles, one of which is less than (or equal to) π and the other of which is greater than (or equal to) π. Suppose that the angle which is less than π does not include the branch cut, and let $\theta_1, \theta_2 \in (-\pi, \pi)$ denote the angles between the lines and the positive real axis; then $|\theta_1 - \theta_2| < \pi$. We may assume that $\theta_1 > \theta_2$ (just reorder if not!). The two rays may be parameterised as

$$\gamma_1(t) = t(\cos\theta_1 + i\sin\theta_1), \qquad t \in (0, +\infty),$$
$$\gamma_2(t) = t(\cos\theta_2 + i\sin\theta_2), \qquad t \in (0, +\infty).$$

Under the square root map, these two rays will be mapped to the rays

$$t \mapsto t^{1/2}\left(\cos\frac{\theta_1}{2} + i\sin\frac{\theta_1}{2}\right), \qquad t \in (0, +\infty),$$
$$t \mapsto t^{1/2}\left(\cos\frac{\theta_2}{2} + i\sin\frac{\theta_2}{2}\right), \qquad t \in (0, +\infty),$$

which make an angle of $\frac{1}{2}(\theta_1 - \theta_2)$ with each other – i.e., *half* that of the angle between the original two rays. This does not contradict what we know about the relationship between analytic and conformal maps, though, since the square root function is not analytic at the origin.

Marking: 2 marks for showing (this requires some kind of computation, not simply a picture) that the angle between the image lines is half that between the original lines. (The requirement 'make an angle of less than π with each other' was actually an accidental red herring.) 1 mark for saying that this is not a contradiction, 1 mark for an explanation as to why.