MAT334, COMPLEX VARIABLES, SUMMER 2020. PROBLEMS FOR MAY 25 – 29

Due Tuesday, June 2, at 12:00 noon EDT.

1. [8 marks] Determine $\operatorname{Log} z$ for each of the following points and branch cuts:
(a) $z = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$, branch cut along $\theta = \pi$, interval $(-\pi, \pi)$.
(b) $z = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$, branch cut along $\theta = 0$, interval $(0, 2\pi)$.
(c) $z = e$, branch cut along $\theta = \pi/2$, interval $(\pi/2, 5\pi/2)$.
(d) $z = e$, branch cut along $\theta = \pi$, interval $(-\pi, \pi)$.

2. [5 marks] Compute the following difference of limits:

$$\lim_{\theta \to 0^+} \operatorname{Log} re^{i\theta} - \lim_{\theta \to 2\pi^-} \operatorname{Log} re^{i\theta},$$

where $r > 0$. Does this difference depend on which branch of Log is used? What would happen if we considered instead the difference

$$\lim_{\theta \to \theta_0^+} \operatorname{Log} re^{i\theta} - \lim_{\theta \to (\theta_0 + 2\pi)^-} \operatorname{Log} re^{i\theta}$$

where θ_0 is any real number?

3. [4 marks] For a given complex number z, use the quadratic formula and the relation

$$\cos w = \frac{e^{iw} + e^{-iw}}{2}$$

to compute all complex numbers w satisfying $\cos w = z$.

4. We know that the exponential function e^z is analytic on the entire complex plane, and hence conformal at each point. Let us see what this map looks like in practice.
(a) [6 marks] Consider straight lines parallel to the real and imaginary axes. What is the image of these lines under the map $z \mapsto e^z$? (For example, if you parameterise the two lines as $\gamma_k(t)$, what kind of curve is $e^{\gamma_k(t)}$?) Sketch a couple representative examples (both the original lines and the image curves).
(b) [4 marks] Now consider two lines passing through the origin, making angles θ_1 and θ_2 with the positive real axis. What is the image of these two curves under the map $z \mapsto e^z$? Sketch the image curves for two particular values of θ_1 and θ_2 (neither of which is a multiple of $\pi/2$!).
(c) [3 marks] How does your work from (a) and (b) exemplify the conformality of e^z?

5. We know that branches of root functions are analytic on their domains, and hence conformal there. Let us see how this works out in practice.
(a) [8 marks] Choose a particular branch of the square root function $z \mapsto z^{1/2}$. (Make sure you indicate your choice clearly!) Consider straight rays from the origin and circles centred on the origin; what is their image under this map? Derive formulas and sketch a couple representative examples (sketch both original and image curves). How does this exemplify the conformality of your particular branch of $z \mapsto z^{1/2}$?
(b) [4 marks] Consider two rays from the origin which make an angle of less than π with each other. What is the angle between the images of these lines under the map from (a)? Does this contradict what we know about the relationship between analytic functions and conformal maps? Why or why not?