CONTRACTING THE BOUNDARY OF A RIEMANNIAN 2-DISC

YEVGENY LIOKUMOVICH, ALEXANDER NABUTOVSKY, AND REGINA ROTMAN

Abstract. Let D be a Riemannian 2-disc of area A, diameter d and length of the boundary L. We prove that it is possible to contract the boundary of D through curves of length $\leq L+200d\max\{1,\ln\frac{\sqrt{A}}{d}\}$. This answers a twenty-year old question of S.Frankel and M.Katz, a version of which was asked earlier by M.Gromov.

We also prove that a Riemannian 2-sphere M of diameter d and area A can be swept out by loops based at any prescribed point $p\in M$ of length $\leq 200d\max\{1,\ln\frac{\sqrt{A}}{d}\}$.

1. Main results

Consider a 2-dimesional disc D with a Riemannian metric. M. Gromov asked if there exists a universal constant C, such that the boundary of D could be homotoped to a point through curves of length less than $C\max\{|\partial D|, diam(D)\}$, where $|\partial D|$ denotes the length of the boundary of D and $diam(D)$ denotes its diameter. This question is a Riemannian analog of the well-known (and still open) problem in geometric group theory asking about the relationship between the filling length and filling diameter (see [Gr93]).

S. Frankel and M. Katz answered the question posed by Gromov negatively in [FK]. They demonstrated that there is no upper bound for lengths of curves in an "optimal" homotopy contracting ∂D in terms of $|\partial D|$ and $diam(D)$. Then they asked if there exists such an upper bound if one is allowed to use the area $Area(D)$ of D in addition to $|\partial D|$ and $diam(D)$. In this paper we will prove that the answer for this question is positive, and, moreover, provide nearly optimal upper bounds for lengths of curves in an "optimal" contracting homotopy in terms of $|\partial D|, diam(D)$ and $Area(D)$. Note that S. Gersten and T. Riley ([GerR]) proved a similarly looking result in the context of geometric group theory. Yet in the Riemannian setting their approach seems to yield an upper bound with the leading terms $const(|\partial D| + diam(D)\max\{1,\ln\frac{\sqrt{Area(D)}}{inj(D)}\})$, where $inj(D)$ denotes the injectivity radius of the disc, and so does not lead to a solution of the problem posed by Frankel and Katz.

Define the *homotopy excess*, $exc(D)$, of a Riemannian disc D as the infimum of x such that for every $p\in\partial D$ the boundary of D is contractible to p via loops of length $\leq |\partial D|+x$ based at p. Let $exc(d,A)$ denote the supremum of $exc(D)$ over all discs D of area $\leq A$ and diameter $\leq d$. The examples of [FK] imply the existence of a positive constant $const$ such that $exc(d,A) \geq const\ d \max\{1, \ln \frac{\sqrt{A}}{d}\}$. The first of our main results implies that this lower bound is optimal up to a constant factor:

Main Theorem A.

$$exc(d,A) \leq 200d \max\{1,\ \ln \frac{\sqrt{A}}{d}\}$$

In fact, we are able to prove that $\limsup_{\frac{\sqrt{A}}{d} \longrightarrow \infty} \frac{exc(d,A)}{d \ln \frac{\sqrt{A}}{d}} \leq \frac{12}{\ln \frac{3}{2}} < 30$ (see the remark at after the proof of Theorem 1.2 in section 7). On the other hand, analysing the examples of Frankel and Katz we were able to prove that $\liminf_{\frac{\sqrt{A}}{d} \longrightarrow \infty} \frac{exc(d,A)}{d \ln \frac{\sqrt{A}}{d}} \geq \frac{1}{8 \ln 2} > 0.18$ and believe that the same examples could be used to get a better lower bound $\frac{1}{2 \ln 2}$.

Here are some other upper estimates:

Theorem 1.1. *For any Riemannian 2-disc D and a point $p \in \partial D$ there exists a homotopy γ_t of loops based at p with $\gamma_0 = \partial D$ and $\gamma_1 = \{p\}$, such that*

$$|\gamma_t| \leq 2|\partial D| + 686\sqrt{Area(D)} + 2diam(D)$$

for all $t \in [0,1]$.

It easy to see that any upper bound for the lengths of $|\gamma_t|$ should be greater than $2diam(D)$. Therefore the upper bound provided by Theorem 1.1 is optimal for fixed values of $Area(D)$ and $|\partial D|$, when $diam(D) \longrightarrow \infty$. However, the next theorem provides a better bound, when $Area(D) \longrightarrow \infty$ or $|\partial D| \longrightarrow \infty$ and immediately implies Main Theorem A stated above.

Theorem 1.2. *For any Riemannian 2-disc D and a point $p \in \partial D$ there exists a homotopy γ_t of loops based at p with $\gamma_0 = \partial D$ and $\gamma_1 = \{p\}$, such that*

$$|\gamma_t| \leq |\partial D| + 159diam(D) + 40diam(D) \max\{0, \ln \frac{\sqrt{Area(D)}}{diam(D)}\}$$

for all $t \in [0,1]$.

As a consequence of the previous theorems we obtain related results about diastoles of Riemannian 2-spheres M. A *diastole* of M was defined by F. Balacheff and S. Sabourau in [BS] as

$$dias(M) = \inf_{(\gamma_t)} \sup_{0 \leq t \leq 1} |\gamma_t|$$

where (γ_t) runs over families of *free loops* sweeping-out M. More precisely,the family (γ_t) corresponds to a generator of $\pi_1(\Lambda M, \Lambda^0 M)$, where ΛM denotes the space of free loops on M and $\Lambda^0 M$ denotes the space of constant loops.

In [S, Remark 4.10] S.Sabourau gave an example of Reimannain two-spheres with arbitrarily large ratio $\frac{dias(M_n)}{\sqrt{Area(M_n)}}$. In [L] the first author gave an example of Riemannian two-spheres M_n with arbitrarily large ratio $\frac{dias(M_n)}{diam(M_n)}$. We show that if both diameter and area of M are bounded, the diastole can not approach infinity. Moreover, for every $p \in M$ one can define $Bdias_p(M)$ by the formula

$$Bdias_p(M) = \inf_{(\gamma_t)} \sup_{0 \leq t \leq 1} |\gamma_t|$$

where (γ_t) runs over families of *loops based at* p sweeping-out M. Now define the *base-point diastole* $Bdias(M)$ as $\sup_{p \in M} Bdias_p(M)$. It is clear that $Bdias(M) \geq dias(M)$. We prove the following inequalities:

Theorem 1.3. (Main Theorem B.) *For any Riemannian 2-sphere M we have*

$$A. \quad Bdias(M) \leq 664\sqrt{Area(M)} + 2diam(M);$$

$$B. \quad Bdias(M) \leq 159diam(M) + 40diam(M)\max\{0, \ln\frac{\sqrt{Area(M)}}{diam(M)}\}.$$

Moreover, as $\frac{diam(M)}{\sqrt{Area(M)}} \longrightarrow 0$,

$$Bdias(M) \leq (\frac{12}{\ln\frac{3}{2}} + o(1))diam(M)\ln\frac{\sqrt{Area(M)}}{diam(M)}.$$

We noticed that one can modify the examples from [FK] to construct a sequence of Riemannian metrics on S^2 such that $\frac{diam}{\sqrt{Area}} \longrightarrow \infty$ but $Bdias \geq 2diam + const\sqrt{Area}$ for an absolute positive constant *const*. (A formal proof involves ideas from [L] and will appear elsewhere. The resulting Riemannian 2-spheres look like very thin ellipsoids of rotation with a disc near one of its poles replaced by a Frankel-Katz 2-disc with area that is much larger than the area of the ellipsoid.) Combining this observation with inequality A we see that $Bdias(M) = 2diam(M) + O(\sqrt{Area(M)})$, when $\frac{\sqrt{Area(M)}}{diam(M)} \longrightarrow 0$, and the dependence on $Area(M)$ in $O(\sqrt{Area(M)})$ cannot be improved. One can also use the examples in [FK] (as well as the ideas from [L]) to demonstrate that inequality B provides an estimate for $Bdias(M)$, which is optimal up to a constant factor, when $\frac{diam(M)}{\sqrt{Area(M)}} \longrightarrow 0$.

Note that in [BS] F.Balacheff and S.Sabourau show that if 1-parameter families of loops in the definition of the diastole are replaced with 1-parameter families of one-cycles, then for every Riemannian surface Σ of genus g the resulting homological diastole $dias_Z(\Sigma)$ satisfies

$$dias_Z(\Sigma) \leq 10^8(g+1)\sqrt{Area(\Sigma)}.$$

The proof of Theorem 1.1 will proceed by first considering subdiscs of D of small area and small boundary length and then obtaining the general result for larger and larger subdiscs by induction. The parameter of the induction will be $\lfloor \log_{\frac{4}{3}} \frac{Area\ D'}{\epsilon(D)} \rfloor$, where D' denotes a (variable) subdisc and $\epsilon(D) > 0$ is very small. As it is the case with many inductive arguments, it is more convenient to prove a stronger statement. To state this stronger version of Theorem 1.1 we will need the following notations:

Definition 1.4. For each $p \in D$ $d_p(D) = max\{dist(p,x)|x \in D\}$. Let $d_D = max\{d_p(D)|p \in \partial D\}$.

From the definition we see that $d_D \leq diam(D)$.

If l_1 and l_2 are two non-intersecting simple paths between points p and q of D, then $l_1 \cup -l_2$ is a simple closed curve bounding a disc $D' \subset D$. We will show that there exists a path homotopy from l_1 to l_2 such that the lengths of the paths in this homotopy are bounded in terms of area, diameter and length of the boundary of D'.

Definition 1.5. Let D be a Riemannian disc and $D' \subset D$ be a subdisc. Define a *relative path diastole* of D' as

$$pdias(D', D) = \sup_{p,q \in \partial D'} \inf_{(\gamma_t)} \sup_{t \in [0,1]} |\gamma_t|$$

where (γ_t) runs over all families of paths from p to q $\gamma_t : [0,1] \to D$ with $\gamma_t(0) = p$, $\gamma_t(1) = q$, where $\gamma_0 = l_1$ and $\gamma_1 = l_2$ are subarcs of $\partial D' = l_1 \cup -l_2$ intersecting only at their endpoints p, q. Let $pdias(D) = pdias(D, D)$.

Theorem 1.6. *A. For any Riemannian 2-disc D with $|\partial D| \leq 2\sqrt{3}\sqrt{Area(D)}$*

$$pdias(D) \leq |\partial D| + 664\sqrt{Area(D)} + 2d_D.$$

B. For any Riemannian 2-disc D with $|\partial D| \leq 6\sqrt{Area(D)}$

$$pdias(D) \leq |\partial D| + 686\sqrt{Area(D)} + 2d_D.$$

C. For any Riemannian 2-disc D with $|\partial D| > 6\sqrt{Area(D)}$

$$pdias(D) \leq |\partial D| + 2\lceil \log_{\frac{4}{3}}(\frac{|\partial D| - 4\sqrt{Area(D)}}{2\sqrt{Area(D)}}) \rceil \sqrt{Area(D)} + 686\sqrt{Area(D)} + 2d_D$$

$$\leq 2|\partial D| + 686\sqrt{Area(D)} + 2d_D.$$

D. Also, if $d \geq 3\sqrt{A}$,

$$exc(d, A) \leq 3d + \frac{2}{\ln\frac{4}{3}}\sqrt{A}\ln(\frac{2}{3}(\frac{2d}{\sqrt{A}} - 4)) + 686\sqrt{A}.$$

Of course, Theorem 1.1 immediately follows from Theorem 1.6 C. The second inequality in Part C of the theorem can be easily proven by observing that $\frac{2\ln(\frac{2(x-4)}{3})}{(\ln\frac{4}{3})x} < 0.9735 < 1$ for $x \in [6, \infty]$. Setting $x = \frac{|\partial D|}{\sqrt{Area(D)}}$ we obtain the desired inequality. The last inequality provides a much better upper bound for $exc(d, A)$, when $\sqrt{A} << d$ and implies that $\lim_{\frac{\sqrt{A}}{d}\longrightarrow 0} exc(d, A) \leq 3d$.

Open problem. Is it true that, when d is fixed, and $A \longrightarrow 0$, $exc(d, A) = 2d + O(\sqrt{A})$?

Here is the plan of the rest of the paper. In the next section we recall Besicovich theorem and use it to reduce Theorem 1.6 A-C to proving (slightly stronger) estimates for subdiscs of D, where the length of the boundary does not exceed $6\sqrt{Area(D)}$. In the same section we apply this result to prove the desired assertion for subdiscs of D with area bounded by a very small constant. At the beginning of section 3 we review a result by P. Papasoglu ([P]) asserting that for every Riemannian 2-sphere S and every ϵ there exists a simple closed curve of length $\leq 2\sqrt{3}\sqrt{Area(S)} + \epsilon$ that divides the sphere into two domains with areas not less than $\frac{1}{4}Area(S)$ and not greater than $\frac{3}{4}Area(S)$. Then we prove an analogous result for Riemannian 2-discs. Section 4 contains two auxilliary results about a relationship of d_D and $d_{D'}$ for a subdisc D' of D. Section 5 contains the proof of the main Theorem 1.6 A-C (and, thus, Theorem 1.1). In section 7 we deduce Theorem 1.2 from Theorem 1.1. Here the key intermediate result is that an arbitrary Riemannian 2-disc D can be subdivided into two subdiscs with areas in the interval $[\frac{1}{3}Area(D) - \epsilon^2, \frac{2}{3}Area(D) + \epsilon^2]$ by a simple curve of length $\leq 2diam(D) + 2\epsilon$ connecting two points of ∂D, where ϵ can be made arbitrarily small. The proof of this result will be given in section 6. It uses a modification of Gromov's filling technique and is reminiscent to a proof of a version of the result of Papasoglu quoted above presented by F. Balacheff and S. Sabourau in [BS]. In section 7 we also prove Theorem 1.6 D. At the end of section 7 we explain how Theorem 1.3 follows from Theorems 1.1 and 1.2.

2. Besicovitch Lemma and reduction to the case of curves with short boundaries

The main tool of this paper is the following theorem due to A.S.Besicovitch [B] (see also [BBI] and [Gr99] for generalizations and many applications of this theorem).

Theorem 2.1. *Let D be a Riemannian 2-disc. Consider a subdivision of ∂D into four consecutive subarcs (with disjoint interiors) $\partial D = a \cup b \cup c \cup d$. Let l_1 denote the length of a minimizing geodesic between a and c; l_2 denote the length of a minimizing geodesic between b and d. Then*

$$Area(D) \geq |l_1||l_2|$$

In this section we use Besicovitch lemma to prove two lemmae. Lemma 2.2 implies that the second inequality of Theorem 1.6 follows from the first. Lemma 2.3 says that boundaries of small subdiscs of D can be contracted through short curves.

Lemma 2.2. ***(Reduction to Short Boundary Case)*** *Let ϵ_0, C be any non-negative real numbers.*
A. Suppose that $|\partial D| > 6\sqrt{Area(D)}$ and that for all subdiscs $D' \subset D$ satisfying $|\partial D'| \leq 6\sqrt{Area(D)}$ we have $pdias(D', D) \leq (1+\epsilon_0)|\partial D'| + C\sqrt{Area(D)} + 2d_{D'}$. Then

$$pdias(D) \leq (1+\epsilon_0)|\partial D| + 2\lceil \log_{\frac{4}{3}}(\frac{|\partial D| - 4\sqrt{Area(D)}}{2\sqrt{Area(D)}})\rceil\sqrt{Area(D)} + C\sqrt{Area(D)} + 2d_D.$$

B. Assume that D is contained in a disc D_0, and all subdiscs $D' \subset D$ satisfying $|\partial D'| \leq 6\sqrt{Area(D)}$ satisfy $pdias(D', D_0) \leq (1+\epsilon_0)|\partial D'| + C\sqrt{Area(D)} + 2d_{D'}$. Then

$$pdias(D, D_0) \leq (1+\epsilon_0)|\partial D| + 2\lceil \log_{\frac{4}{3}}(\frac{|\partial D| - 4\sqrt{Area(D)}}{2\sqrt{Area(D)}})\rceil\sqrt{Area(D)} + C\sqrt{Area(D)} + 2d_D.$$

Proof. A. First, we are going to prove A. For each subdisc $D' \subset D$ define

$$n(D') = \log_{\frac{4}{3}}(\frac{|\partial D'| - 4\sqrt{Area(D)}}{2\sqrt{Area(D)}})$$

For each $n \in \{0, ..., \lceil n(D)\rceil\}$ (where $\lceil$ x $\rceil$ denotes the integer part of x+1) and every subdisc $D' \subset D$ with $n - 1 < n(D') \leq n$ we will show that $pdias(D', D) \leq (1+\epsilon_0)|\partial D'| + 2n\sqrt{Area(D)} + C\sqrt{Area(D)} + 2d_{D'}$

For $n = 0$ we have $|\partial D'| \leq 6\sqrt{Area(D)}$ so we are done by assumption in the statement of the theorem.

Suppose the conclusion is true for all integers smaller than n. Let $p, q \in \partial D'$. Let l_1 and l_2 be two subarcs of $\partial D'$ from p to q, $|l_2| \leq |l_1|$. We will construct a homotopy of paths from l_1 to l_2 of length $\leq (1+\epsilon_0)(|l_1| + |l_2|) + (C + 2n)\sqrt{Area(D')} + 2d_{D'}$.

Subdivide $l_1 \cup -l_2$ into four arcs a_1, a_2, a_3 and a_4 of equal length so that the center of a_2 coincides with the center of l_2. By Besicovitch lemma there exists a curve α between opposite sides a_1 and a_3 or a_2 and a_4 of length $\leq \sqrt{Area(D')}$.

We have two cases.

Case 1. Both endpoints t_1 and t_2 of α belong to the same arc l_i ($i = 1$ or 2). Denote the arc of l_i between t_1 and t_2 by β. Note that $\frac{1}{4}(|l_1| + |l_2|) \leq |\beta| \leq \frac{3}{4}(|l_1| + |l_2|)$. In particular, the disc D_1 bounded by $\alpha \cup -\beta$ has boundary of length $\leq \frac{3}{4}|\partial D'| + \sqrt{Area(D')} \leq (4 + 2(\frac{4}{3})^{n-1})\sqrt{Area(D)}$. The induction assumption implies that $pdias(D_1, D) \leq (1+\epsilon_0)|\partial D_1| + (C + 2n - 2)\sqrt{Area(D)} + 2d_{D_1}$.

We claim that $d_{D_1} \leq d_{D'} + \frac{1}{2}\sqrt{Area(D')}$. Indeed, let $y \in \partial D_1$. If $y \in \partial D'$ then the geodesic from y to x does not cross α as both are minimizing geodesics, hence the distance in D_1 $d_{D_1}(y,x) \leq d_{D'}$. If $x \in \alpha$ then the triangle inequality implies that $d_{D_1}(y,x) \leq \frac{1}{2}|\alpha| + d_{D'}$.

Hence, for an arbitrarily small $\delta > 0$ we can homotop l_i to $pt_1 \cup \alpha \cup t_2 q$ through curves of length

$$\leq |l_i \setminus \beta| + (1+\epsilon_0)|\partial D_1| + (C+2n-2)\sqrt{Area(D)} + 2d_{D_1} + \delta$$

$$\leq (1+\epsilon_0)|\partial D'| + \sqrt{Area(D)} + (C+2n-2)\sqrt{Area(D)} + 2d_D + \sqrt{Area(D)} + \delta.$$

Now consider the disc D_2 bounded by $pt_1 \cup \alpha \cup t_2 q \cup -l_j$, where l_j $(j \neq i)$ is the other arc. As in the case of D_1, we can homotop $pt_1 \cup \alpha \cup t_2 q$ to l_j through curves of length $\leq (1+\epsilon_0)|\partial D'| + 2n\sqrt{Area(D)} + C\sqrt{Area(D)} + 2d_{D'}$.

Case 2. $t_1 \in l_1$, $t_2 \in l_2$. Let β_i denote the subarc of l_i from p to t_i and σ_i denote the subarc of l_i from t_i to q. Consider the subdisc $D_1 \subset D'$ bounded by $\beta_1 \cup \alpha \cup -\beta_2$. As in Case 1 the inequality $|\partial D_1| \leq \frac{3}{4}|\partial D'| + \sqrt{Area(D')}$ combined with the induction assumption implies that $pdias(D_1, D) \leq (1+\epsilon_0)|\partial D_1| + (C+2n-2)\sqrt{Area(D)} + 2d_{D_1}$. Using the estimate $d_{D_1} \leq d_{D'} + \frac{1}{2}\sqrt{Area(D')}$ we can homotop l_1 to $\beta_2 \cup -\alpha \cup \sigma_1$ through curves of length

$$\leq (1+\epsilon_0)|\partial D'| + (2n+C)\sqrt{Area(D)} + 2d_{D'} + \delta.$$

In exactly the same way we homotop $\beta_2 \cup -\alpha \cup \sigma_1$ to l_2 using the inductive assumption for the other disc $D_2 = D' \setminus D_1$.

This proves that $pdias(D) \leq (1+\epsilon_0)|\partial D| + 2\lceil n(D) \rceil \sqrt{Area(D)} + C\sqrt{Area(D)} + 2d_D$.

This completes the proof of A. The proof of its relative verion B is almost identical to the proof of A.

□

Lemma 2.3. ***(Small Area)*** *Given a positive ϵ_0 there exists a positive ϵ, such that if $D \subset D_0$ with $Area(D) < \epsilon$, then*

$$pdias(D, D_0) \leq (1+\epsilon_0)|\partial D|,$$

when $|\partial D| \leq 6\sqrt{\epsilon}$, and

$$pdias(D, D_0) \leq (1+\epsilon_0)|\partial D| + 2\lceil \log_{\frac{3}{4}}(\frac{|\partial D| - 4\sqrt{Area(D)}}{2\sqrt{Area(D)}}) \rceil \sqrt{Area(D)} + 2d_D,$$

when $|\partial D| > 6\sqrt{\epsilon}$.

Proof. Lemma 2.2 B implies that in order to prove the second inequality it is enough to find $\epsilon > 0$ such that for all subdiscs D' of D with $|\partial D'| \leq 6\sqrt{\epsilon}$

$$pdias(D', D_0) \leq (1+\epsilon_0)|\partial D'| + 2d_{D'}.$$

For all sufficiently small radii r every ball $B_r(p) \subset D$ is bilipschitz homeomorphic to a convex subset of the positive half-plane $\mathbb{R}^2_+$ with bilipshitz constant $L = 1 + O(r^2)$.

Hence, for a sufficiently small ϵ if $|\partial D'| \leq 6\sqrt{\epsilon}$, then $pdias(D', D_0) \leq (1 + O(\epsilon))pdias(U, V)$, where $U \subset V \subset \mathbb{R}^2_+$, $|\partial U| \leq (1 + O(\epsilon))|\partial D'|$ and V is convex. We will show that $pdias(U, V) \leq |\partial U|$ thereby proving the result.

Let $p, q \in \partial U$ and $l_1 : [0,1] \to V$, $l_2 : [0,1] \to V$ be two arcs of ∂U from p to q. Let $\alpha^i_t : [0,1] \to V$ denote a parametrized straight line from p to $l_i(t)$. We define a homotopy of paths from l_1 to l_2 as $\gamma_t = \alpha^1_{2t} \cup l_1|_{[2t,1]}$ for $0 \leq t \leq \frac{1}{2}$ and $\gamma_t = \alpha^2_{2-2t} \cup l_2|_{[2-2t,1]}$. We have $|\gamma_t| \leq max(|l_1|, |l_2|) \leq |\partial U|$.

Now we can choose $\epsilon > 0$ so that $(1 + O(\epsilon))pdias(U, V) \leq (1 + \epsilon_0)pdiast(U, V)$, and the desired assertion follows. □

Remark. Note that it is not difficult to prove the existence of $\epsilon > 0$ such that for each disc $D \subset D_0$ of area $\leq \epsilon$ one has $pdias(D, D_0) \leq |\partial D|$.Yet the proof is more complicated than the proof above. Moreover, this strengthening of Lemma 2.3 does not lead to any improvements of our main estimates. Therefore, we decided to state Lemma 2.3 only in its weaker form.

3. Subdivision by short curves

The following theorem was proven by P. Papasoglu in [P]. For the sake of completeness we will present a proof which is a slightly simplified version of the proof given by Papasoglu.

Theorem 3.1. ***(Sphere Subdivision)*** *Let $M = (S^2, g)$ be a Riemannian sphere. For every $\delta > 0$ there exists a simple closed curve γ subdividing M into two discs D_1 and D_2, such that $\frac{1}{4}Area(M) \leq Area(D_i) \leq \frac{3}{4}Area(M)$ and $|\gamma| \leq 2\sqrt{3}\sqrt{Area(M)} + \delta$*

Proof. Consider the set S of all simple closed curves on M dividing M into two subdiscs each of area $\geq \frac{1}{4}Area(M)$. To see that this set is non-empty one can take a level set of a Morse function on M and connect its components by geodesics. From arcs of these geodesics one can obtain paths between components of the level set that can be made disjoint by a small perturbation. Traversing each of the connecting paths twice one obtains a closed curve that becomes simple after a small perturbation.

Choose a positive ϵ. Let $\gamma \in S$ be a curve that is $\epsilon-$minimal. (In other words, its length is greater than or equal to $\inf_{\tau \in S} |\tau| + \epsilon$.) Let D be one of the two discs forming $M \setminus \gamma$ that has area $\geq \frac{1}{2}Area(M)$. If we subdivide γ into four equal arcs then by Besicovitch Lemma there is a curve α connecting two opposite arcs of length $\leq \frac{\sqrt{3}}{2}\sqrt{A}$. Observe that α subdivides D into two discs, and at least one of these discs has area $\geq \frac{1}{4}Area(M)$. Hence, the boundary of this disc is an element of S of length $\leq \frac{3}{4}|\gamma| + |\alpha|$. By $\epsilon-$minimality of γ we must have

$$|\gamma| \leq \frac{3}{4}|\gamma| + \frac{\sqrt{3}}{2}\sqrt{A} + \epsilon.$$

Therefore, $|\gamma| \leq 2\sqrt{3}\sqrt{A} + 4\epsilon$. □

Our next result is an analog of the previous result for 2-discs.

Proposition 3.2. ***(Disc Subdivision Lemma)*** *Let D be a Riemannian 2-disc. For any $\delta > 0$ there exists a subdisc $\overline{D} \subset D$ satisfying*

(1) $\frac{1}{4}Area(D) - \delta^2 \leq Area(\overline{D}) \leq \frac{3}{4}Area(D) + \delta^2$

(2) $|\partial\overline{D} \setminus \partial D| \leq 2\sqrt{3}\sqrt{Area(D)} + \delta$

Proof. Without any loss of generality we can assume $\delta \leq \frac{\sqrt{Area(D)}}{10}$. Attach a disc D' of area $\leq \delta^2$ to the boundary of D so that $M = D' \cup D$ is a sphere of area $\leq Area(D) + \delta^2$. We apply Theorem 3.1 to M to obtain a close curve γ of length $\leq 2\sqrt{3}\sqrt{Area(D)} + \delta$ that divides D into two subdiscs D_1 and D_2 with areas in the interval $[\frac{1}{4}Area(D) - \delta^2, \frac{3}{4}Area(D) + \delta^2]$. Without any loss of generality we can assume that either γ does not intersect $|\partial D|$ or intersects it transversally. (Note that the idea of attaching a disc of a very small area to the boundary of D and applying Theorem 3.1 appears in [BS].)

If $\gamma \cap \partial D$ is empty then $D_i \subset D$ for one of D_i's and setting $\overline{D} = D_i$ we obtain the desired result.

A more difficult case arises when $\gamma \cap \partial D \neq \oslash$. For each $i = 1, 2$ $D_i \cap D$ may have several connected components. Those components, D^j, are subdiscs of D of area $\leq \frac{3}{4}Area(D) + \delta^2$. If the area of one of them is $\geq \frac{1}{4}Area(D) - \delta$, then we can choose this subdisc as $\overline{D}$, and we are done. Otherwise, we can start erasing connected components of $\gamma \bigcap D$ one by one. When we erase a connected component of $\gamma \bigcap D$, the two subdiscs adjacent to the erased arc merge into a larger subdisc of area $\leq \frac{1}{2}Area(A) - 2\delta^2$. We continue this process until we obtain a new subdisc of area $\geq \frac{1}{4}Area(A) - \delta^2$, and choose this subdisc as $\overline{D}$. □

4. Bounds for $d_{D'}$.

We will also need the following lemmae relating d_D with $d_{D'}$ for a subdisc D' of D.

Lemma 4.1. *Let $D' \subset D$ be a subdisc, $p \in \partial D$ and $p' \in \partial D'$ be two points connected by a minimizing geodesic α in D. Then $d_{D'} + |\alpha| \leq d_D + |\partial D'|$*

Proof. Let β be a minimizing geodesic in D' from a point on the boundary to a point $x \in D'$, s.t. $|\beta| = d_{D'}$ (It exists by compactness). Let γ be a minimizing geodesic from p to x. Denote by γ_1 the arc of γ from p to the point where it first intersects $\partial D'$ and by γ_2 the arc from the point where it last intersects $\partial D'$ to x. Then by triangle inequality

$$|\alpha| \leq |\gamma_1| + \frac{1}{2}|\partial D'|,$$
$$|\beta| \leq |\gamma_2| + \frac{1}{2}|\partial D'|.$$

Hence, $d_{D'} + |\alpha| \leq d_D + |\partial D'|$. □

Lemma 4.2. *Suppose $D' \subset D$ is a subdisc with $\partial D' \cap \partial D$ non-empty. Then $d_{D'} \leq d_D + |\partial D' \setminus \partial D|$.*

Proof. Note that $\partial D' \setminus \partial D$ is a colection of countably many open arcs with endpoints on ∂D.

Let β be a minimizing geodesic in D' from a point $p \in \partial D'$ to a point $x \in D'$, such that $|\beta| = d_{D'}$. Let α be a minimizing geodesic in D from p to x.

We will construct a new curve α' which agrees with α on the interior of D' and lies entirely in the closed disc D'. If α does not intersect any arcs of $\partial D' \setminus \partial D$ we set $\alpha' = \alpha$. Otherwise, let a_1 denote the first arc of $\partial D' \setminus \partial D$ intersected by α. Let p_1 (resp. q_1) denote the point where α intersects a_1 for the first (resp. last) time. (If $p \in \partial D' \setminus \partial D$, then $p_1 = p$.) We replace the arc of α from p_1 to q_1 with the subarc of a_1. We call this new curve α_1. We find the next (after a_1) arc $a_2 \subset \partial D' \setminus \partial D$ that α_1 intersects and replace a subarc of α_1 with a subarc of a_2. We continue this process inductively until we obtain a curve $\alpha' = \alpha_n$ that lies in D'.

Note that $|\beta| \leq |\alpha'| \leq |\alpha| + |\partial D' \setminus \partial D|$. Hence, if $p \in \partial D$, then $|\alpha| \leq d_D$ and we are done.

If p belongs to an arc $a \subset \partial D' \setminus \partial D$, then let a' be a subarc of a connecting p to a point of ∂D, such that $a' \cap \alpha' = \{p\}$. (Note, that in this case $p = p_1$.) Then $|\alpha'| + |a'| \leq |\alpha| + |\partial D' \setminus \partial D|$. □

5. Proof of Theorem 1.1 A-C.

We are now ready to prove statements A to C of Theorem 1.6.

Let ϵ_0 be an arbitrary positive number less than 0.001. Fix an $\epsilon = \epsilon(\epsilon_0) > 0$ small enough for Lemma 2.3.

Let N be an integer defined by

$$(\frac{4}{3})^{N-1}\epsilon \leq Area(D) < (\frac{4}{3})^N\epsilon$$

Let $\delta < \min\{\epsilon, (\frac{4}{3})^N\epsilon - Area(D)\}$. For each $n \in \{0, 1, ..., N\}$ and for every subdisc $D' \subset D$ with $(\frac{4}{3})^{n-1}\epsilon - \frac{\delta}{2^{N-n+1}} \leq Area(D') < (\frac{4}{3})^n\epsilon - \frac{\delta}{2^{N-n}}$ we will show

A. If $|\partial D'| \leq 2\sqrt{3}\sqrt{Area(D)}$ then

$$pdias(D', D) \leq |\partial D'| + 664\sqrt{Area(D')} + 2d_{D'}.$$

B. If $|\partial D'| \leq 6\sqrt{Area(D)}$ then

$$pdias(D', D) \leq |\partial D'| + 686\sqrt{Area(D')} + 2d_{D'}.$$

C. If $|\partial D'| > 6\sqrt{Area(D)}$ then

$$pdias(D', D) \leq (1+\epsilon_0)|\partial D'| + 2\lceil \log_{\frac{4}{3}}(\frac{|\partial D'| - 4\sqrt{Area(D')}}{2\sqrt{Area(D')}})\rceil\sqrt{Area(D')} + 686\sqrt{Area(D')} + 2d_{D'}$$

$$\leq 2|\partial D'| + 686\sqrt{Area(D')} + 2d_{D'}.$$

Passing to the limit as $\epsilon_0 \longrightarrow 0$, we will obtain the assertion of the theorem.

For $n = 0$ we have $Area(D') \leq \epsilon$, and so by Lemma 2.3 we are done. Assume the result holds for every integer less than n. By Lemma 2.2 statement C can be reduced to the following statement:

C'. For every subdisc $D'' \subset D'$ such that $|\partial D''| \leq 6\sqrt{Area(D')}$ we have

$$pdias(D'', D) \leq |\partial D''| + 686\sqrt{Area(D'')} + 2d_{D''}.$$

In particular this imples statement B. We will be proving C' sometimes making special considerations for the case $|\partial D''| \leq 2\sqrt{3}\sqrt{Area(D')}$, which will imply statement A.

For any $p, q \in \partial D''$ we will construct a homotopy between the two arcs satisfying this bound.

Let l_1 and l_2 be two arcs of $\partial D''$ connecting p and q. Let $\overline{D} \subset D''$ by a subdisc satisfying the conclusions of Proposition 3.2 with δ equal to our current δ divided by 2^{N+2}.

We have two cases.

Case 1. $\partial\overline{D} \cap \partial D''$ is nonempty. Then $\partial\overline{D} \setminus \partial D''$ is a collection of arcs $\{a_i\}$. For each arc a_i we have a corresponding subdisc $D_i \subset D'' \setminus \overline{D}$ with $a_i \subset \partial D_i$ and $Area(D_i) \leq \frac{3}{4}Area(D'') + \frac{\delta}{2^{N-n+2}} < (\frac{4}{3})^{n-1}\epsilon - \frac{\delta}{2^{N-n+1}} \leq Area(D')$.

If $l_1^i = l_1 \cap \partial D_i$ is a non-empty arc, we use the inductive assumption to define a path homotopy of l_1^i to $\partial D_i \setminus l_1^i$ through curves of length

$$\leq 2|\partial D_i| + (\frac{\sqrt{3}}{2}686 + 4\sqrt{3})\sqrt{Area(D'')} + 2d_{D''} + O(\delta),$$

$$\leq |\partial D_i| + 686\sqrt{Area(D'')} + 2d_{D''} + O(\delta),$$

where we have used Lemma 4.2 to bound d_{D_i}.

This procedure homotopes l_1 to a curve $l \subset l_2 \cup \partial\overline{D}$. Now using the inductive assumption for $\overline{D}$ we continue our homotopy from l_1 to l_2 without exceeding the length bound. (At this stage we get rid of $\bar{D}$.) At the end of this stage it remains only to homotope arcs on $\partial\bar{D}$ to corresponding arcs of l_2 through some of the discs D_i. This step is similar to the already described step involving arcs of l_1.

Virtually the same argument proves statement A for this case.

Note that diameter term d_D is not used in an essential way in this case. Its necessity comes from Case 2.

Case 2. $\partial\overline{D}$ does not intersect $\partial D''$. Denote $\partial\overline{D}$ by γ. $D'' \setminus \gamma$ is the union of an annulus A and an open disc $\overline{D}$. Let α_1 (resp. α_2) be a minimizing geodesic from p (resp. q) to γ. Let γ_i denote the arc of γ, such that $l_i \cup \alpha_2 \cup -\gamma_i \cup -\alpha_1$ bounds a disc D_i whose interior is in the annulus A. Note that $Area(D_i) \leq \frac{3}{4}Area(D'') + O(\delta)$.

Proposition 5.1. *A. If $|l_i| \leq 2\sqrt{Area(D')} + O(\delta)$ then there is a homotopy from l_i to $\alpha_1 \cup \gamma_i \cup -\alpha_2$ through curves of length $\leq 664\sqrt{Area(D')} + 2d_{D''} + O(\delta)$.*

B. If $2\sqrt{Area(D')} < |l_i| \leq 6\sqrt{Area(D')} + O(\delta)$ then there is a homotopy from l_i to $\alpha_1 \cup \gamma_i \cup -\alpha_2$ through curves of length $\leq 686\sqrt{Area(D')} + 2d_{D''} + O(\delta)$.

To prove Proposition 5.1 we will need the following lemma.

Lemma 5.2. *If $|\partial D_i| > M = \max\{10\sqrt{3}\sqrt{Area(D')}, 4|l_i| + 2\sqrt{3}\sqrt{Area(D')}\} + O(\delta)$, then there exists a geodesic β of length $\leq \frac{\sqrt{3}}{2}\sqrt{Area(D)} + O(\delta)$ connecting α_1 to α_2 such that the endpoints of β divide ∂D_i into two arcs of length $\leq \frac{3}{4}|\partial D_i|$.*

Proof. We subdivide ∂D_i into 4 equal subarcs, starting from point p. By Besicovitch lemma we can connect two opposite arcs by a curve β of length $\leq \frac{\sqrt{3}}{2}\sqrt{Area(D'')} + O(\delta)$. Now we consider different cases.

Suppose first that β connects a point of α_k ($k = 1$ or 2) with another point of α_k. Since α_k is length minimizing we obtain $\frac{1}{4}|\partial D_i| \leq \frac{\sqrt{3}}{2}\sqrt{Area(D'')} + O(\delta)$ so $|\partial D_i| \leq 2\sqrt{3}\sqrt{Area(D')} + O(\delta)$.

If β connects a point of l_i to another point of l_i then $|l_i| \geq \frac{1}{4}|\partial D_i|$. Similiarly, if β connects two points of γ_i then $|\partial D_i| \leq 8\sqrt{3}\sqrt{Area(D'')} + O(\delta)$.

Suppose β connects a point of l_i to a point of γ_i. Since α_1 and α_2 are length minimizing, we must have $|\alpha_1| + |\alpha_2| \leq |l_i| + 2|\beta|$, so $|\partial D_i| \leq 2|l_i| + 2|\beta| + |\gamma_i| \leq 2|l_i| + 3\sqrt{3}\sqrt{Area(D')}$.

Suppose β connects a point x of γ_i and a point y of α_k. Since α_i is a geodesic minimizing distance to the curve γ, we conclude that the subarc of α_i between y and $\gamma_i$has length $\leq |\beta|$. Hence, $\frac{1}{4}|\partial D_i| \leq |\gamma_i| + |\beta|$, so $|\partial D_i| \leq 10\sqrt{3}\sqrt{Area(D')} + O(\delta)$.

Now, suppose β connects a point of l_i and a point of α_k. Then $\frac{1}{4}|\partial D_i| \leq |l_i| + |\beta|$ yielding $|\partial D_i| \leq 4|l_i| + 2\sqrt{3}\sqrt{Area(D')}$.

If $|l_i| \leq 2\sqrt{3}\sqrt{Area(D')}$, then in all of the above cases we have $|\partial D_i| \leq 10\sqrt{3}\sqrt{Area(D')}+O(\delta)$. If $|l_i| > 2\sqrt{3}\sqrt{Area(D')}$, then $|\partial D_i| \leq 4|l_i| + 2\sqrt{3}\sqrt{Area(D')}$.

The only remaining case is when β connects α_1 to α_2. □

Proof of Proposition 5.1. Proof of B. Suppose first that $|\partial D_i| \leq M$.

Hence, since $Area(D_i) \leq \frac{3}{4}Area(D'') + \frac{\delta}{2^{N-n+2}}$ and using the inductive assumption we can homotope l_i to $\alpha_1 \cup \gamma_i \cup -\alpha_2$ through curves of length

$$\leq (2+\epsilon_0)|\partial D_i| + 686\sqrt{Area(D_i)} + 2d_{D_i} + O(\delta).$$

Note that since $|l_i| \leq 6\sqrt{Area(D')} + O(\delta)$, we have $M \leq (24+2\sqrt{3})\sqrt{Area(D')} + O(\delta)$ and $M - |l_i| \leq \max\{10\sqrt{3}\sqrt{Area(D')}, 3|l_i| + 2\sqrt{3}\sqrt{Area(D')}\} + O(\delta) \leq (18 + 2\sqrt{3})\sqrt{Area(D')} + O(\delta)$.

Therefore, using Lemma 4.2 the lengths of curves in the homotopy are bounded by

$$\leq |\partial D''| + (18+2\sqrt{3}+24+2\sqrt{3}+\frac{\sqrt{3}}{2}686+2(18+2\sqrt{3}))\sqrt{Area(D')} + 2d_{D''} + O(\delta)$$

$$< |\partial D''| + 686\sqrt{Area(D')} + 2d_{D''}$$

Note that our choice of the constant $686 > (78+8\sqrt{3})/(1-\frac{\sqrt{3}}{2})$ is motivated by the last of these inequalities.

Now consider the case, when $|\partial D_i| > M$. Lemma 5.2 implies that we can subdivide D_i into two subdiscs D_i^1 and D_i^2 of boundary length $\leq \frac{3}{4}|\partial D_i| + \frac{\sqrt{3}}{2}\sqrt{Area(D')} + O(\delta)$ by a curve β_1 connecting α_1 and α_2. For each of subdiscs D_i^j we have an argument completely analogous to that of Lemma 5.2. We apply it repeatedly until we obtain a sequence of discs D^k stacked on top of each other with $|\partial D^1| \leq M$ and $|\partial D^k| \leq (10\sqrt{3})\sqrt{Area(D')} + O(\delta)$ for $k \geq 2$. The discs are separated by Besicovitch geodesics $\{\beta^k\}$. Let α_1^k (α_2^k) denote the subarcs of α_1 (α_2) between p (resp. q) and the endpoint of β^k.

We homotope l_i to $\alpha_1^1 \cup \beta^1 \cup -\alpha_2^1$ as described above. Then we homotope $\alpha_1^k \cup \beta^k \cup -\alpha_2^k$ to $\alpha_1^{k+1} \cup \beta^{k+1} \cup -\alpha_2^{k+1}$ using the inductive assumption in disc D^{k+1} through curves of length

$$\leq (2+\epsilon_0)|\partial D^{k+1}| + 686\sqrt{Area(D^{k+1})} + 2d_{D^{k+1}} + |\alpha_1| + |\alpha_2|$$

$$\leq ((40+10\epsilon_0)\sqrt{3}+\frac{\sqrt{3}}{2}686)\sqrt{Area(D')}+2d_{D''}+O(\delta) < 686\sqrt{Area(D')}+2d_{D''}+O(\delta),$$

where we have used Lemma 4.1 to bound $2d_{D^{k+1}} + |\alpha_1| + |\alpha_2|$.

The proof of A is analogous with the only difference that both M and $M - |l_i|$ are majorized by $\leq 10\sqrt{3}\sqrt{Area(D')}$. The only purpose of A is to obtain a somewhat better value of the constant at $\sqrt{Area(D)}$ in Theorems 1.3 A and Theorem 1.6 A. Therefore we omit the details.

This finishes the proof of Propostion 5.1.

Using Propostion 5.1 we homotope l_1 to $\alpha_1 \cup \gamma_1 \cup -\alpha_2$. Using inductive assumption in the disc $\overline{D}$ and Lemma 4.1 we homotop $\alpha_1 \cup \gamma_1 \cup -\alpha_2$ to $\alpha \cup \gamma_2 \cup -\alpha_2$. By applying Proposition 5.1 again we homotope $\alpha \cup \gamma_2 \cup -\alpha_2$ to l_2. This finishes the proof of statements A to C of Theorem 1.6. The proof of statement D is presented in the last section.

6. Subdivision by short curves II.

In this section we are going to prove the following theorem:

Theorem 6.1. *A. Let M be a Riemannian 2-sphere, p a point in M. For every positive ϵ there exists a simple based loop on M of length $\leq 2\max_{x\in M} dist(x,p) + \epsilon$ based at p that divides M into two discs with areas in the interval $(\frac{2}{3}Area(M) - \epsilon, \frac{2}{3}Area(M) + \epsilon)$.*

B. Let D be a Riemannian 2-disc. For every $\epsilon > 0$ there exists a curve β of length $\leq 2\delta_D + \epsilon$ with endpoints on the boundary ∂D, which does not self-intersect and divides D into subdiscs D_1 and D_2 satisfying

$$\frac{1}{3}Area(D) - \epsilon^2 \leq Area(D_i) \leq \frac{2}{3}Area(D) + \epsilon^2$$

Proof. A. Fix a diffeomorphism $f : S^2 \longrightarrow M$. Consider a very fine triangulation of S^2. We are assuming that the length of the image of each 1-simplex of this triangulation under f does not exceed ϵ, and the area of the image of each 2-simplex does not exceed ϵ^2. Extend this triangulation to a triangulation of D^3 constructed as the cone of the chosen triangulation of S^2 with one extra vertex v at the center. We are going to prove the assertion by contradiction. Assume that all simple loops of length $\leq 2d + \epsilon$ based at p divide M into two subdiscs one of which has area $\leq \frac{1}{3}Area(D) - \epsilon^2$. We are going to construct a continuous extension of f to D^3 obtaining the desired contradiction. We are going to map the center v of D into p. We are going to map each 1-simplex $[vv_i]$ of the considered triangulation of D^3 to a shortest geodesic connecting p with $f(v_i)$. We extend f to all 2-simplices $[vv_iv_j]$ by contracting the loop formed by the shortest geodesic connecting p, $f(v_i)$ and $f(v_j)$ within one of two discs in M bounded by this loop that has a smaller area. This disc has area $\leq \frac{1}{3}Area(D) - \epsilon^2$. Now it remains to construct the extension of f to the interiors of all 3 simplices $[vv_iv_jv_k]$ of the chosen triangulation of D^3. Note that the area of the image of the boundary of this simplex does not exceed $3(\frac{1}{3}Area(M) - \epsilon^2) + \epsilon^2 < Area(M)$.

Therefore the restriction of the already constructed extension of f to this boundary has degree zero, and, therefore, is contractible. This completes our extension process and yields the desired contradiction.

B. We can deduce B from the proof of A by collapsing ∂D into a point p and repeating the argument used to prove part A for the resulting (singular) 2-sphere. Yet one can give another direct proof by contradiction as follows. Assume that the assertion of the theorem is false. Consider a very fine geodesic triangulation of the disc. Assume that the areas of all triangles are less than ϵ^2. We are going to construct a retraction f of D onto ∂D, thereby obtaining a contradiction as follows: First we are going to map all new vertices of the triangulation. Each vertex will be mapped to (one of) the closed points on ∂D. Each edge v_iv_j will be mapped to one of two arcs in ∂D connecting $f(v_i)$ with $f(v_j)$. We have two possible choices. We choose the arc that together with the geodesic broken line $f(v_i)v_iv_jf(v_j)$ encloses a subdisc D_{ij} of D of a smaller area (which is $\leq \frac{1}{3}Area(D)-\epsilon^2$). Now we need to extend the constructed map to all triangles $v_iv_jv_k$ of the triangulation. We claim that the chosen arcs between $f(v_i), f(v_j)$ and $f(v_k)$ do not cover ∂D, and therefore $f(\partial v_iv_jv_k)$ can be contracted within ∂D yielding the desired contradiction. Indeed, otherwise the discs D_{ij}, D_{ik} and D_{jk} would cover all D with a possible exception of a part of the triangle $v_jv_jv_k$. But this is impossible as the sum of their areas does not exceed $Area(D)-3\epsilon^2$ which is strictly less than $Area(D)-\epsilon^2$.

7. Proofs of Theorem 1.2, 1.3 and 1.6 D

Definition 7.1. For each disc D define δ_D by the formula $\delta_D = \sup_{x\in D} dist(x, \partial D)$.

Note the following properties of δ_D:
1. $d_D - \frac{|\partial D|}{2} \leq \delta_D \leq d_D$.
2. If $D' \subset D$ then $\delta_{D'} \leq \delta_D$.

Lemma 7.2. *For each $n \geq 1$ $pdias(D) \leq 2|\partial D| + 2d_D + 8n\delta_D + 686\sqrt{(\frac{2}{3})^n Area(D)}$.*

Proof. The proof is by induction on n. If $n = 0$, then Theorem 1.1 implies that $pdias(D) \leq 2|\partial D| + 2d_D + 686\sqrt{Area(D)}$

Suppose the claim is true for $n-1$. Choose $\epsilon > 0$ that can later be made arbitrarily small. We use Theorem 6.1 to subdivide D into two subdiscs of area $\leq \frac{2}{3}Area(D)+\epsilon^2$ by a curve β of length $\leq 2\delta_D + \epsilon^2$.

The inductive assumption implies that we can homotope an arc of l_1 (from the definition of *pdias*) over each of D_i via curves of length less than or equal to $2|\partial D| + 2|\beta| + 2d_{D_i} + 8(n-1)\delta_{D_i} + 686\sqrt{(\frac{2}{3})^{n-1}Area(D_i)} + O(\epsilon)$.

We have $d_{D_i} \leq d_D + |\beta|$ by Lemma 4.2. Hence the lengths of the curves are bounded by $2|\partial D| + 8n\delta_D + 2d_D + 686\sqrt{(\frac{2}{3})^n Area(D)} + O(\epsilon)$. □

The next proposition allows us to get rid of the extra $|\partial D|$ in our estimates.

Proposition 7.3. *Suppose that $f(x,y,z)$ is a continuous function such that $pdias(D) \leq f(|\partial D|, diam(D), Area(D))$ for every disc D. Then*

$$pdias(D) \leq \max_{0\leq t\leq |\partial D|} |\partial D| - t + f(\min\{2(|\partial D|-t), 2t, 2diam(D)\}, diam(D), Area(D))$$

Proof. Let p, q be endpoints of $l_1 \cup -l_2 = \partial D$ and β be a minimizing geodesic from p to q. We will construct a homotopy from l_1 to β. We choose a small $\epsilon > 0$ and partition $[0,1]$ by $N+1$ points $\{0 = a_0, ..., a_N = 1\}$ so that $|l_1([a_i, a_{i+1}])| \leq \epsilon$. Let α_i denote a minimizing geodesic from p to $l_1(a_i)$. Inductively we homotop $\alpha_i \cup l_1([a_i, 1])$ to $\alpha_{i+1} \cup l_1([a_{i+1}, 1])$.

Consider the subdisc bounded by $\partial D^i = \alpha_i \cup l_1([a_i, a_{i+1}]) \cup -\alpha_{i+1}$. Since α_i are length minimizing we have $|\partial D^i| \leq \min\{2(|\partial D| - t) + \epsilon, 2t + \epsilon, 2diam(D)\} + \epsilon$, where $t = |l_1([0, a_i])|$. Using our assumption we obtain a homotopy from $\alpha_i \cup l_1([a_i, 1])$ to $\alpha_{i+1} \cup l_1([a_{i+1}, 1])$. The homotopy between β and l_2 can be constructed in the same way. It remains to pass to the limit as $\epsilon \longrightarrow 0$. □

In particular, we can now prove statement D of Theorem 1.6. From statements B, C we know that

$$pdias(D) \leq |\partial D| + 2\max\{0, \lceil \log_{\frac{4}{3}}(\frac{|\partial D| - 4\sqrt{Area(D)}}{2\sqrt{Area(D)}})\rceil\}\sqrt{Area(D)} + 686\sqrt{Area(D)} + 2d_D.$$

Then if we set $L_t = \min\{2(|\partial D| - t), 2t, 2diam(D)\}$ we obtain an estimate

$$pdias(D) < \max_t(|\partial D| - t + \frac{L_t}{2}) + \frac{L_t}{2} + 2\max\{0, \lceil \log_{\frac{4}{3}}(\frac{L_t - 4\sqrt{Area(D)}}{2\sqrt{Area(D)}})\rceil\}\sqrt{Area(D)}$$

$$+686\sqrt{Area(D)} + 2diam(D)$$

$$\leq |\partial D| + 2\max\{0, \lceil \log_{\frac{4}{3}}(\frac{diam(D)}{\sqrt{Area(D)}} - 2\rceil)\}\sqrt{Area(D)} + 686\sqrt{Area(D)} + 3diam(D),$$

as $L_t \leq 2diam(D)$ and $-t + \frac{L_t}{2} \leq 0$. The formula for excess follows from this estimate.

An analogous coarser estimate that uses Theorem 1.1 instead of Theorem 1.6 C yields

$$pdias(D) \leq |\partial D| + 5diam(D) + 686\sqrt{Area(D)}.$$

Proof of Theorem 1.2

Using Lemma 7.2 we obtain

$pdias(D) \le |\partial D| + (5+8n)diamD + 686\sqrt{(\frac{2}{3})^n Area(D)} + O(\epsilon)$

Let k be any positive number, such that $n = 2\log_{3/2}(\frac{\sqrt{Area(D)}}{diam(D)k})$ is a natural number. Then the previous estimate can be written as

$$pdias(D) \le |\partial D| + (686k + 5 + 16\log_{3/2}(\frac{1}{k}) + 16\log_{3/2}(\frac{\sqrt{Area(D)}}{diam(D)}))diam(D)$$

Suppose first that $\sqrt{Area(D)} > (\frac{2}{3})^{6.5} diam(D)$. Note that for some $k \in [(\frac{2}{3})^7, (\frac{2}{3})^{6.5}]$ we will have $2\log_{3/2}(\frac{\sqrt{Area(D)}}{diam(D)k}) \in \mathbb{N}$. It is easy to check that for each k in this interval $686k + 16\log_{3/2}(\frac{1}{k}) < 154$. Hence, from the previous inequality and using $\frac{16}{\ln(3/2)} < 40$ we obtain

$$pdias(D) \le |\partial D| + 159diam(D) + 40\ln(\frac{\sqrt{Area(D)}}{diam(D)})diam(D).$$

If $\sqrt{Area(D)} \le (\frac{2}{3})^{6.5} diam(D)$, then

$$pdias(D) \le |\partial D| + 5diam(D) + 686\sqrt{Area(D)} \le |\partial D| + 50diam(D).$$

Remark. We can obtain a better asymptotic estimate if instead of a bound with $2|\partial D|$ we use the one from Theorem 1.6 with the logarithmic term. Then for $\frac{\sqrt{Area(D)}}{diam(D)} \to \infty$ we obtain $pdias(D) < |\partial D| + (\frac{12}{\ln\frac{3}{2}} + o(1))\ln(\frac{\sqrt{Area(D)}}{diam(D)})diam(D)$. (Note that $\frac{12}{\ln\frac{3}{2}} = 29.5956\ldots$).

Note that the 25 percent improvement of the constant at $diam(D)\ln\frac{\sqrt{Area(D)}}{diam(D)}$ (from $\frac{16}{\ln\frac{3}{2}}$ to $\frac{12}{\ln\frac{3}{2}}$) comes from the fact that the term $2|\beta|$ in the proof of Lemma 7.2 can be replaced by $|\beta|$,and $8n\delta_D$ in the right hand side in the inequality of Lemma 7.2 becomes $6n\delta_D$.

Proof of Theorem 1.3.

Let p be an arbitrary point of M. Take the metric ball $B_\epsilon(p)$ of a very small radius ϵ centered at p and choose a point $q \in \partial B_\epsilon(p)$. Applying Theorem 1.6 A we see that one can contract $\partial B_\epsilon(p)$ in $M \setminus B_\epsilon(p)$ as a loop based at q via loops of length not exceeding the right hand side in Theorem 1.3 A plus $O(\epsilon)$. Now we can attach two copies of the geodesic segment (pq) connecting p and q at the beginning and the end of each of those loops based at q. As the result, we will obtain a family of loops based at p. Finally, add a family of loops based at p that constitutes a homotopy between the constant loop p and $(pq) * \partial B_\epsilon(p) * (qp)$ and a family of loops that contracts $(pq) * (qp)$ over itself to the constant loop p. The lengths of all these new loops are $O(\epsilon)$. As the result, we obtain a family of loops based at p of lengths $\leq 664\sqrt{Area(M)} + 2diam(M) + O(\epsilon)$ that sweeps-out M. Now pass to the limit as $\epsilon \longrightarrow 0$.

To prove the inequality B we can proceed as above with the only difference that $\partial B_\epsilon(p)$ will be contracted in $M \setminus B_\epsilon(p)$ using Theorem 1.2 instead of Theorem 1.6 A. Finally, note that, when $\frac{\sqrt{Area(M)}}{diam(M)} \longrightarrow \infty$, one can improve the constant in inequality B exactly as it had been described in the remark after the proof of Theorem 1.2 above. The result will be the last assertion in Theorem 1.2. $\square$

References

[B] A. S. Besicovitch, "On two problems of Loewner", J. London Math. Soc. 27, (1952). 141-144

[BS] F. Balacheff, S. Sabourau, "Diastolic and isoperimetric inequalities on surfaces", Ann. Sci. Ecole Norm. Sup. 43 (2010) 579-605.

[BBI] D. Burago, Yu. Burago, S. Ivanov, "A Course in Metric Geometry", AMS, 2001.

[FK] S. Frankel, M. Katz, "The Morse landscape of a Riemannian disc", Annales de l'Inst. Fourier 43 (1993), no. 2, 503-507.

[GerR] S. Gersten, T. Riley, "Filling length in finitely presentable groups", Geom. Dedicata, 92(2002), 41-58.

[Gr93] M.Gromov, "Asymptotic invariants of infinite groups", in "Geometric Group Theory", v. 2, 1-295, London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993.

[Gr99] M.Gromov, "Metric Structures for Riemannian and Non-Riemannian Spaces", Birkhauser, 1999.

[L] Y. Liokumovich, "Spheres of small diameter with long sweep-outs", preprint, http://arxiv.org/abs/1105.6159, to appear in Proceedings of the AMS.

[P] P.Papasoglu, "Cheeger constants of surfaces and isoperimetric inequalities", Trans. Amer. Math. Soc 361 (2009), no. 10, 5139-5162.

[S] S.Sabourau, "Filling radius and short closed geodesics of the 2-sphere", Bull. Soc. Math. France 132 (2004), 105-136.

Department of Mathematics, University of Toronto, Toronto, Canada
E-mail address: `e.liokumovich@utoronto.ca`

Department of Mathematics, University of Toronto, Toronto, Canada
E-mail address: `alex@math.toronto.edu`

Department of Mathematics, University of Toronto, Toronto, Canada
E-mail address: `rina@math.toronto.edu`