MULTIPLIER SPECTRUM OF MAPS ON $\mathbb{P}^1$: THEOREMS OF JI-XIE AND JI-XIE-ZHANG

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ABSTRACT. Zhuchao Ji and Junyi Xie recently proved that the multiplier spectrum of a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ will uniquely determine the map up to conjugacy, for a general choice of f (i.e., for f in a Zariski open subset of the space of all maps of degree d, in each degree $d > 1$). These notes present a sketch of the proof of the main result in [JX3], which itself builds on the proofs in [JX1] and the main theorem of [JX2]. In the final section, we provide a simplification of their proof and discuss a theorem of Ji, Xie, and Zhang from [JXZ]. These notes were prepared for lectures at Harvard and Toronto in November 2023 and edited in Spring 2024.

1. OVERVIEW

Fix a degree $d \geq 2$. Let M_d denote the moduli space of maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ over $\mathbb{C}$ of degree d, so

$$\mathrm{M}_d = \mathrm{Rat}_d / \operatorname{Aut} \hat{\mathbb{C}}$$

where Rat_d is the space of all rational functions $f(z) = P(z)/Q(z)$ of degree d, where the polynomials $P(z), Q(z) \in \mathbb{C}[z]$ have no common factors, and $\operatorname{Aut} \hat{\mathbb{C}} \simeq \mathrm{PSL}_2\mathbb{C}$ is the group of Möbius transformations acting by conjugation; that is, $\varphi \cdot f = \varphi \circ f \circ \varphi^{-1}$. The space M_d is an affine algebraic variety (with singularities) of dimension $2d - 2$.

Milnor [Mi1] showed that $\mathrm{M}_2 \simeq \mathbb{C}^2$; Silverman [Si2] constructed the moduli spaces M_d (and compactifications) over $\operatorname{Spec} \mathbb{Z}$ using geometric invariant theory; Levy [Le2] proved that M_d is rational for all degrees $d \geq 2$; and West [We] provided an explicit description of M_3 as a subvariety of a certain weighted projective space. Bergeron, Filom, and Nariman [BFN] recently studied the topology of M_d. Other than these results, we know little about M_d as an algebraic variety; in particular, we do not have a good dynamical description of the ring $\mathbb{C}[\mathrm{M}_d]$ of regular functions on M_d.

The **multipliers** of a map f define natural functions on M_d, as follows. Recall that the multiplier of $f \in \mathrm{Rat}_d$ at a periodic point z_0 with $f^n(z_0) = z_0$ is the derivative $(f^n)'(z_0)$ (in coordinates where $z_0 \neq \infty$). The multipliers are invariant under the conjugacy action.

For each $n \in \mathbb{N}$, we can associate to f a vector in $\mathbb{C}^{d^n+1}$ by taking the symmetric functions in the multipliers at all points fixed by f^n, counted with multiplicity. In this way, we define morphisms

$$\tau_{d,n} : \mathrm{M}_d \longrightarrow \mathbb{C}^{d+1} \times \mathbb{C}^{d^2+1} \times \cdots \times \mathbb{C}^{d^n+1},$$

recording multipliers for all periods $\leq n$. Note that there is a great deal of redundancy in the definition of $\tau_{d,n}$, and there are known relations among the multipliers (that need not concern us for the purposes of this note). Two maps $f, g \in \mathrm{Rat}_d$ are said to be **isospectral** if they have the same image under $\tau_{d,n}$ for all n.

McMullen [Mc1] proved that the multipliers of a map f determine its conjugacy class $[f]$ in M_d up to finitely many choices, as long as f is not a flexible Lattès map. That is, for each fixed degree d and for all sufficiently large periods n (depending on d), the restriction

$$\tau_{d,n} : \mathrm{M}_d \setminus \mathcal{L} \longrightarrow \mathbb{C}^{N(d,n)}$$

has finite fibers, where $\mathcal{L}$ is the algebraic curve of flexible Lattès maps (which exists only when $d = m^2$ for some integer $m \geq 2$; it has two components in each such degree). The multipliers are constant along the components of $\mathcal{L}$.

Ji and Xie have recently proved that, by choosing n large enough (possibly depending on the degree d) and by restricting $\tau_{d,n}$ to a smaller but Zariski open dense subset of M_d, each fiber of $\tau_{d,n}$ will consist of a single point. In other words:

Theorem 1.1. [JX3] *In each degree $d \geq 2$, the multiplier map $\tau_{d,n}$ is generically one-to-one, for all $n \gg 0$.*

This result was already known for $d = 2$, as Milnor's isomorphism $\mathrm{M}_2 \simeq \mathbb{C}^2$ is by $\tau_{d,1}$ [Mi1]. And in degree $d = 3$, it was known that $\tau_{d,2}$ is generically one-to-one [We, Got, HT]. The proof of Theorem 1.1 in [JX3] requires $d \geq 4$ (to apply a result of Pakovich [Pa2] in one step).

Remark 1.2. Note that $\tau_{d,n}$ may not be injective on all of $\mathrm{M}_d \setminus \mathcal{L}$ for any choice of n. There are two known constructions of isospectral maps that are not conjugate and not in $\mathcal{L}$:

- rigid Lattès maps in degree d from non-isomorphic CM elliptic curves equipped with the same endomorphism (e.g., multiplication by $\sqrt{-d}$ when d is not a square); and
- compositions of the form $f_1 \circ f_2$ and $f_2 \circ f_1$ in composite degrees.

It is not known if there are other examples of non-conjugate isospectral maps. See [Mi3] for more on Lattès maps and [Pa1] for more about compositional equivalence.

Remark 1.3. It is not hard to show that $\tau_{d,2}$ is generically finite in every degree $d \geq 2$ [Ge]. Indeed, one marks the $d^2 + 1$ points of period 1 and 2 at $f(z) = z^d$

and computes that the rank of the multiplier map is maximal. See also [Gor]. The following conjecture is stated explicitly in [JX3]:

Conjecture 1.4. *In every degree $d \geq 2$, the period-2 multiplier map $\tau_{d,2}$ is generically injective.*

Remark 1.5. When working with the subspace MPoly_d of polynomial maps in M_d, there are many additional tools and certain arguments simplify. Note that the dimension of MPoly_d is $d-1$. Restricted to MPoly_d, the fixed-point multiplier map $\tau_{d,1}$ is generically finite in all degrees d, with degree $(d-2)!$ [Fu], and the explicit count of $\tau_{d,1}^{-1}(\lambda)$ in MPoly_d is known for all $\lambda \in \mathbb{C}^{d+1}$ [Su, CP].

2. PROOF STRATEGY IN [JX3]

In this section, we sketch the proof of Theorem 1.1 in [JX3]. We fix a degree $d \geq 4$ and argue by contradiction.

Suppose the generic degree of $\tau_{d,n}$ is not 1 for any n. Choose n_d large enough so that $\tau_{d,n_d}(f) = \tau_{d,n_d}(g)$ if and only if f and g are isospectral. Let $\tau := \tau_{d,n_d}$, and let κ be its generic degree. Let U_d be a Zariski open subset of M_d on which τ is a covering map from U_d to its image, of topological degree equal to κ.

In particular, τ induces a κ-fold symmetry of U_d (we can build an algebraic correspondence over U_d that identifies points in fibers of τ). Moreover, recalling that J-stability is characterized by properties of the multipliers [MSS, Ly] (see also [Mc2, Chapter 4]), τ induces a κ-fold symmetry of the bifurcation locus within U_d. Furthermore, as the multipliers also determine the Lyapunov exponent $L(f)$ of a map f on $\mathbb{P}^1$ (with respect to its measure of maximal entropy μ_f) [Be], we also obtain a κ-fold symmetry of the bifurcation current $T_{\mathrm{bif}} := dd^c L$ [De1, De2] on U_d.

Remark 2.1. To obtain the symmetry of the bifurcation current, Ji and Xie take a different approach, by first restricting to special algebraic curves within U_d and then appealing to an arithmetic equidistribution result of [YZ2]. Working directly with the Lyapunov exponent function seems most direct.

Ji and Xie show this extra symmetry cannot exist: in particular, they observe that τ induces a symmetry of the postcritically finite maps along a special algebraic curve in U_d and use the theory developed in their article [JX1] to obtain a contradiction. We present their argument in two parts.

Remark 2.2. A map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is **postcritically finite**, or **PCF**, if each of its critical points has a finite forward orbit. In [JXZ, Theorem 1.12], the authors characterize the PCF maps by their multiplier spectrum, among all maps defined over $\overline{\mathbb{Q}}$. This characterization is *not* needed for the Ji-Xie proof of Theorem 1.1, but knowing this is possible helps put their proof in context.

2.1. **Part 1: choosing a good 1-parameter family.** We begin by appealing to recent work of Pakovich in [Pa2]. By assuming $d \geq 4$ and shrinking U_d if necessary, we can assume that the maps f with conjugacy class in U_d have no automorphisms and no nontrivial "partners"; that is, for any $f \in \mathrm{Rat}_d$ representing a point of U_d and any other $g \in \mathrm{Rat}_d$, if the map $(f, g) : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ has a periodic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ which is neither vertical nor horizontal, then g is conjugate to f and the curve is one of the obvious ones (e.g., the graph of an iterate) [Pa2, Theorem 1.6]. In particular, we obtain the following important fact: *for any f with conjugacy class in U_d, and for any g which is not conjugate to f, the invariant curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of the product map (f, g) consist only of unions of vertical and horizontal lines.*

We will work with a special 1-parameter family of maps in U_d, defined by requiring that $2d - 3$ of the critical points (i.e., all but one) are periodic, and each has a distinct period. Such curves are Zariski dense in M_d [JX3, Lemma 2.5] (or see the proof of [De3, Theorem A]), so we can be sure there exists such a curve C in the open set U_d and so that τ is κ-to-one on this (possibly reducible) curve (by replacing C with $\tau^{-1}(\tau(C))$ if needed).

Now pass to a branched cover $X \to C$, if needed, so that we can parameterize this family of maps by f_t, for t in a quasi-projective algebraic curve X, with its remaining free critical point defining a holomorphic map $c : X \to \mathbb{P}^1$. As c is not persistently preperiodic, and the family is not isotrivial, the pair (f, c) must be bifurcating along each irreducible component of X [DF, Theorem 2.5]; that is, the sequence of functions $\{t \mapsto f_t^n(c(t))\}_n$ fails to be a normal family on any component of X. The bifurcation current, defined above as $T_{\mathrm{bif}} = dd^c L$, restricts to X as a measure that can be alternatively defined by

$$\mu_{f,c} = \pi_* \left(\hat{T}_f \wedge [\Gamma_c] \right)$$

where $\pi : X \times \mathbb{P}^1 \to X$ is the projection, $\hat{T}_f$ is the dynamical Green current on $X \times \mathbb{P}^1$ associated to $f : X \times \mathbb{P}^1 \to X \times \mathbb{P}^1$ defined by $f(t, x) = (t, f_t(x))$, and Γ_c is the graph of c; see, for example, [DF, §3].

Note that f_t is PCF for infinitely many parameters t in X, as a simple application of Montel's theorem (as c will be preperiodic to repelling cycles of f_t at a dense set of parameters t in its bifurcation locus). Moreover, there are infinitely many parameters where c is periodic (with periods $\to \infty$); see, for example, [DF, Proposition 2.4]. Note also that the multiplier map τ can detect when c is periodic for f, at least if its orbit is distinct from the periodic orbits of the other critical points, because there will be $2d - 2$ multipliers equal to zero. (Recall that $\tau(f_t) = \tau(f_{t'})$ for $t, t' \in X$ if and only if the two maps are isospectral.) As such, the symmetry induced by τ along X also identifies infinitely many κ-tuples of non-conjugate PCF maps in X.

As κ is larger than 1, we can build (at least) *two* families f_t and g_t, passing to a further branched cover of the parameter space X if necessary, with marked critical

points c and c' respectively, which exhibit a *dynamical coincidence*: the two pairs (f, c) and (g, c') have the same bifurcation loci and the same bifurcation measures on X, and there are infinitely many parameters in X at which c and c' are both periodic (and so in particular both f_t and g_t are PCF at these parameters), while f_t and g_t are not conjugate for any $t \in X$. Recall from the choice of the set U_d that the map (f_t, g_t) on $\mathbb{P}^1 \times \mathbb{P}^1$ has no invariant curves other than (unions of) vertical and horizontal lines, for any $t \in X$. We set

$$(2.1) \qquad \mu_{\text{bif}} := \mu_{f,c} = \mu_{g,c'}$$

on X.

2.2. **Part 2: dynamical-parameter similarity to derive a contradiction.** This is the technical heart of the Ji-Xie proof of Theorem 1.1, and we provide only a brief outline of their argument.

As background, first recall that the Mandelbrot set looks like the corresponding Julia set for $f_c(z) = z^2 + c$ at the parameters c where the critical point is preperiodic to a repelling cycle [Ta]. One can build these similarities in other families of maps on $\mathbb{P}^1$, but also in a measure-theoretic sense, between bifurcation currents and measures of maximal entropy on a Julia set (as done in [BE]), and the similarity extends to parameters where the critical point lands in a hyperbolic/expanding set and not just a single cycle (as in [Ga1, AGMV]) or, as utilized in [JX3], even just under the assumption that f is expanding "enough" along the critical orbit, e.g., if satisfying a Collet-Eckmann-type condition (see [PRS] for details on CE-type conditions).

By the construction of our families f and g over X in §2.1, it turns out that μ_{bif}-almost every parameter satisfies "good enough" expansion conditions; details are given in their earlier work [JX1, Section 4]. Their conclusion is that, since $\mu_{\text{bif}} = \mu_{f,c} = \mu_{g,c'}$, we have $\mu_{\text{bif}} \approx \mu_{f_t} \approx \mu_{g_t}$ at μ_{bif}-a.e. parameter t in X, where μ_{f_t} and μ_{g_t} are the measures of maximal entropy. Here we are being intentionally vague to avoid technicalities. The symbol $\approx$ is representing an asymptotic statement: on a sequence of shrinking neighborhoods V_n of t in the parameter space X, the measures $\alpha_n \mu_{\text{bif}}$ (for constants $\alpha_n \to \infty$) will converge weakly to the measure μ_{f_t} on a small open set in $\mathbb{P}^1$ intersecting the Julia set of f_t.

Ji and Xie explore this concept of asymptotic symmetry between f_t and g_t, at μ_{bif}-a.e. parameter t, in [JX1, Section 7]. In particular, they deduce the existence of a holomorphic isomorphism φ between an open set intersecting the Julia set of f_t and an open set intersecting the Julia set of g_t so that $\varphi_* \mu_{f_t} = \mu_{g_t}$. They then appeal to another paper of theirs [JX2, Theorem 1.7] to conclude that *the product map (f_t, g_t) must have an invariant curve in $\mathbb{P}^1 \times \mathbb{P}^1$ which is not a vertical or horizontal line*, where the curve is locally given as the graph of the holomorphic φ. (The results of [JX2] are built upon the theory of holomorphic local symmetries of Julia sets, as initiated in the work of Levin [Le1]; see also [DFG] for a related result proved recently.

To put the conclusion about the invariant curve in context, recall that Levin-Przytycki [LP] proved that if $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ are two non-exceptional maps defined over $\mathbb{C}$ for which $\mu_f = \mu_g$, then they "almost" share an iterate. In particular, one can deduce from the proofs in [LP] that the equality of measures is equivalent to the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ being preperiodic for the product map (f, g); see [MS, Theorem 1.10]. The asymptotic similarity between f_t and g_t led Ji and Xie to a localized version of this result.)

Ji and Xie further observe that, because the measure μ_{bif} has continuous potentials and so cannot be supported on a countable set in $X(\mathbb{C})$, and because their conclusion holds for μ_{bif}-a.e. parameter $t \in X$, it must hold for at least one transcendental parameter t_0; i.e., with $t_0 \in X(\mathbb{C}) \setminus X(\overline{\mathbb{Q}})$. It follows that the invariant curve in $\mathbb{P}^1 \times \mathbb{P}^1$ for (f_{t_0}, g_{t_0}) must exist in a family; that is, the pair (f_t, g_t) will have an invariant curve in $\mathbb{P}^1 \times \mathbb{P}^1$ (which is neither vertical nor horizontal) for all but finitely many parameters $t \in X(\mathbb{C})$. But this contradicts the choice of the good 1-parameter family f from Part 1 and completes the proof of Theorem 1.1.

2.3. **A stronger conclusion.** Note that the authors have proved more than what was needed to obtain their desired contradiction and complete the proof of Theorem 1.1. They needed only find a single parameter $t_0 \in X$ for which (f_{t_0}, g_{t_0}) has an invariant curve in $\mathbb{P}^1 \times \mathbb{P}^1$ to contradict the choice of the family f. (We will exploit this observation to present a simplification of their proof in Section 3.)

Ji and Xie have shown the following:

Theorem 2.3. [JX3, Theorem 3.4] *Suppose that f_t and g_t are non-isotrivial algebraic families of maps of degree $d \geq 2$ parameterized by t in a quasiprojective algebraic curve X. Suppose that neither f nor g is a family of Lattès maps, and assume that there are infinitely many parameters $t \in X$ at which both f_t and g_t are PCF. Then there exists an algebraic family of curves $W_t \subset \mathbb{P}^1 \times \mathbb{P}^1$, neither vertical nor horizontal, parameterized by $t \in X$, which is invariant for (f_t, g_t) for all t.*

And in fact they have shown even more, if one appeals to more of the results in their earlier article [JX1]. The families f and g are defined over $\overline{\mathbb{Q}}$ and the construction of the families (specifically, the PCF coincidence for the two families) implies that we have arithmetic intersection number $\overline{\mathcal{L}}_{f,c} \cdot \overline{\mathcal{L}}_{g,c'} = 0$ on X, where $\overline{\mathcal{L}}_{f,c}$ is the canonical adelically metrized line bundle associated to the pair (f, c); its archimedean curvature distribution is the measure $\mu_{f,c}$ defined above. As a consequence of this vanishing, Ji and Xie deduce in [JX1, Lemma 3.15] that there is a constant C so that

$$\int G_{W_t}(f_t^n(c_t), g_t^n(c_t')) \, d\mu_{\text{bif}}(t) \ \leq \ C$$

for all iterates $n \geq 0$, for any curve family $W = \{W_t\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ for $t \in X$ not persistently containing any iterate of (c, c'), where G_W is a Green function for W, in

the sense that $G_{W_t}(x) \approx -\log \text{dist}(x, W_t)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. In other words, the orbit of (c, c') is – on average – not too close to the curves of W. (Compare to Diophantine conditions on orbits of points for maps on $\mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$, for example in Silverman's [Si1, Theorem E].) But this on-average separation contradicts the proximity of the orbit to an invariant curve from the asymptotic symmetry between f_t and g_t, so it must be that (c, c') is contained in that invariant curve. In other words, they prove:

Theorem 2.4. [JX3, JX1] *Suppose that f_t and g_t are non-isotrivial algebraic families of maps of degree $d \geq 2$ parameterized by t in a quasiprojective algebraic curve X, and suppose that $c, c' : X \to \mathbb{P}^1$ are active critical points for f and g, respectively. Suppose that neither f nor g is a family of Lattès maps, and assume that there are infinitely many parameters $t \in X$ at which both f_t and g_t are PCF. Then there exists an algebraic family of curves $W_t \subset \mathbb{P}^1 \times \mathbb{P}^1$, parameterized by $t \in X$, which is invariant for (f_t, g_t) for all t and such that $(f_t^n(c_t), g_t^n(c_t'))$ is contained in W_t for all t and for all n.*

3. An alternative approach

As observed in §2.3, the proof in [JX3] provides much more than simply a proof of Theorem 1.1. Here we present an alternative approach, by simplifying the work required for the part of their proof which was sketched above in §2.2.

Our proof begins in the same way by contradiction, as described in the beginning of the previous section and by selecting a good 1-parameter family to work with, exactly as we have detailed in §2.1. We assume we have two algebraic families of maps $f = \{f_t\}$ and $g = \{g_t\}$ parameterized by $t \in X$ as in the conclusion of §2.1; note that the families are defined over $\overline{\mathbb{Q}}$.

We first wish to show that there is at least one parameter $t_0 \in X$ at which c is preperiodic to a repelling cycle of f_{t_0} and c' is preperiodic to a repelling cycle for g_{t_0}. As we observed in §2.1, we already know that there are infinitely many parameters $t \in X$ for which c and c' are simultaneously periodic (but then the cycles are superattracting). And we know that there are infinitely many parameters where c is preperiodic to a repelling cycle of f_t because (f, c) is active, and similarly there exists an infinite set of parameters where c' is preperiodic to a repelling cycle of g_t; the aim is to show that at least one of these parameters coincides.

Remark 3.1. For readers familiar with arithmetic equidistribution, as in the first step of the proof of the main result of [JX1] (and all of the cases of that theorem known before, going back to [BD]), one might expect to begin by applying an equidistribution theorem to deduce that all of the PCF parameters for f and for g must coincide. But note that the equidistribution results for quasiprojective varieties – specifically the versions proved by [YZ2] or [Ga2] that can be used in our setting here where X is

not compact – are not yet known to imply that the two height functions on $X(\overline{\mathbb{Q}})$, associated to the pairs (f, c) and (g, c') respectively, are equal. That would be the case when working with heights on *projective* varieties. (In particular, the analog of the arithmetic Hodge Index theorem of [YZ1] is not known to hold in this setting.) So we cannot deduce immediately that all PCF parameters coincide for the two families of maps. We take another approach, which involves less sophisticated machinery, though it still uses arithmetic input.

As mentioned in Remark 2.2, it was proved in [JXZ, Theorem 1.12] that the PCF maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ can be detected – from among all maps defined over $\overline{\mathbb{Q}}$ – by properties of their multipliers. Here we reformulate (and reprove) their result as follows:

Theorem 3.2. *Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^1$ has degree > 1 and is defined over a number field K. Then f is PCF if and only if there is a finite set S of places of K so that every nonzero multiplier λ for f will satisfy $|\lambda|_v \geq 1$ at all places $v \notin S$.*

Proof. Suppose that f is PCF. Then there are only finitely many places of K at which f can have attracting cycles that are not superattracting. Indeed, for any $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over K, there is a finite set S of places (which correspond to primes of $\mathbb{Q}$ bounded by the degree of the map f) so that for any $v \notin S$, a v-adically attracting cycle that is not superattracting must attract a critical point with infinite orbit; see [Mi2, Corollary 14.5] for archimedean places and [BIJL, Theorem 1.5] for the non-archimedean case.

For the converse, suppose that f has an infinite critical orbit. Extend the number field K if necessary, so that all critical points are defined over K. Conjugating by a Möbius transformation defined over K, we may assume that $z_0 = \infty$ is a critical point with infinite orbit. It was proved in [Si1, Theorem 2.2] that for any finite set S of places, the orbit $\{f^n(z_0)\}$ will contain only finitely many S-integers. In particular, for any choice of S, we can always find a place $v \notin S$ and an iterate n so that $|f^n(0)|_v > 1$. In particular, recalling that f has good reduction at all but finitely many places, there is an infinite set $\mathcal{V}$ of places v so that f has good reduction and the residue class of z_0 modulo v is periodic for f mod v. Let $n(v)$ denote the minimal period of this residue class, and note that the collection $\{n(v) : v \in \mathcal{V}\}$ is unbounded.

Now let us change coordinates again so that the critical point with infinite orbit lies at $z_0 = 0$. Suppose that v is a place in $\mathcal{V}$, and assume that $n = n(v)$ is larger than the periods of any periodic critical points for f. So f^n takes the open v-adic unit disk to itself and contains no superattracting periodic points. By Hensel's Lemma – or more precisely, the Weierstress Preparation theorem as formulated in [Be, Chapter 14] – there is a periodic point z_1 for f in the residue class of z_0, with exact period n. Because $(f^n)'(z_0) = 0$, the derivative of f^n at z_1 must be v-adically smaller than 1 in absolute value, so this cycle is attracting. $\square$

Corollary 3.3. *If $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$ are isospectral, and if f is PCF, then g is PCF.*

It follows that, for our families f and g parameterized by $t \in X$ with $\tau(f_t) = \tau(g_t)$ for all t (so that f_t and g_t are isospectral), a map f_t is PCF if and only if g_t is PCF. As the pair (f, c) is bifurcating, we know that there are infinitely many parameters where c is preperiodic to a repelling cycle for f, so c' must be preperiodic for g at these parameters. Since the bifurcation loci in X are the same for (f, c) and for (g, c'), c' must also be preperiodic to repelling cycles at these parameters (because the pair (g, c') would be stable in a neighborhood of a parameter where c' is preperiodic to an attracting cycle, and PCF maps have no neutral cycles [Mi2, Corollary 14.5]). In other words, there are infinitely may parameters t at which *both* c and c' are preperiodic to repelling cycles. We will let $t_0 \in X$ be one of these parameters.

It is convenient to choose our parameter $t_0 \in X$ so that, in addition, the critical point c for f is *transversely* pre-repelling. This means that the graph of some iterate $t \mapsto f_t^n(c(t))$ in $X \times \mathbb{P}^1$ intersects the graph of a repelling periodic point (defined in a neighborhood of t_0 in X) transversely in $X \times \mathbb{P}^1$, over the parameter t_0. Such parameters are dense in the support of the measure μ_{bif} [Du, Theorem 0.1].

We now exploit the existence of the parameter t_0. We will deduce from the equality of bifurcation measures (2.1) that the measures of maximal entropy $\mu_{f_{t_0}}$ and $\mu_{g_{t_0}}$ are related on some open sets intersecting their supports. From this we will conclude that the pair (f_{t_0}, g_{t_0}) has an invariant curve in $\mathbb{P}^1 \times \mathbb{P}^1$ which is not a union of horizontal and vertical lines. This would contradict the original construction of the family f.

As mentioned above in §2.2, measure-theoretic similarities have been built between parameter spaces and dynamical spaces, when a critical point is preperiodic to a repelling cycle; see, for example, [FG, §4.1.4] and also [BE, Ga1, AGMV]. Recall that the critical point c for f is transversely pre-repelling at t_0. Let $t \mapsto R_f(t)$ and $t \mapsto R_g(t)$ denote parameterizations of the repelling cycles for f and g, respectively, in a neighborhood of t_0 that lie in the forward orbits of c and c', respectively, at t_0. Let $q \geq 1$ denote the order of contact of the graph of an iterate $g^{m_0}(c')$ with the graph of R_g near t_0, in $X \times \mathbb{P}^1$. Choose integer $p \geq 1$ so that R_f and R_g are fixed by f^p and g^p, respectively. Let u be the local potential in X for μ_{bif}, defined by evaluating the homogeneous escape-rate function for f or for g along a lift of the critical point and chosen so that $u(t_0) = 0$. Let $\lambda_f = (f_{t_0}^p)'(R_{f,t_0})$ and let λ_g be a q-th root of the multiplier of g_{t_0} at $R_g(t_0)$. Suppose that φ_f is a linearizing coordinate for f, so that $f_{t_0}^p(\varphi_f(w)) = \varphi_f(\lambda_f w)$ for w in a small disk around 0; similarly for g. Then, as in [FG, Proposition 4.13], we have

$$\lim_{n \to \infty} d^{n_0 + np} u(\lambda_f^{-n}(t - t_0)) = \beta \, U_{f_{t_0}}(\varphi_f(t - t_0))$$

and

$$\lim_{n \to \infty} d^{m_0+np} u(\lambda_g^{-n}(t - t_0)) = \beta' \, U_{g_{t_0}}(\varphi_g((t - t_0)^q))$$

uniformly on a small neighborhood of t_0, where $U_{f_{t_0}}$ and $U_{g_{t_0}}$ denote locally-defined potentials for the measures of maximal entropy (defined by evaluating the homogeneous escape-rate function along a holomorphic section), having value 0 the repelling point, and β and β' are a positive constants. [Caution: the arguments in [FG] were written only for polynomials, but the arguments go through exactly the same.] It follows that the Laplacians of these subharmonic functions converge weakly. Note that, a priori, the two multipliers λ_f and λ_g^q are unrelated, but since they correspond to a rescaling rate at which the same measure μ_{bif} converges to (possibly two different) nontrivial and nonatomic measures, it must be that there is convergence of the sequence $(\lambda_f/\lambda_g)^n$ to some $\alpha \in \mathbb{C}^*$, after passing to a subsequence if necessary; compare [JX1, Proposition 7.2]. We deduce that $h(z) := \varphi_g((\varphi_f^{-1}(z)/\alpha)^q)$ defines a holomorphic map from a small neighborhood of $R_f(t_0)$ in $\mathbb{P}^1$ to a small neighborhood of $R_g(t_0)$ so that $h^*\mu_{g_{t_0}}$ is proportional to $\mu_{f_{t_0}}$. Note that this h is q-to-1. But we can now apply a theorem of Dujardin, Favre, and Gauthier [DFG, Theorem A] to deduce that the graph of h in $\mathbb{P}^1 \times \mathbb{P}^1$ is part of an algebraic curve which is preperiodic for (f_{t_0}, g_{t_0}). This completes the proof.

References

[AGMV] Matthieu Astorg, Thomas Gauthier, Nicolae Mihalache, and Gabriel Vigny. Collet, Eckmann and the bifurcation measure. *Invent. Math.* **217**(2019), 749–797.

[BD] Matthew Baker and Laura DeMarco. Preperiodic points and unlikely intersections. *Duke Math. J.* **159**(2011), 1–29.

[BIJL] Robert Benedetto, Patrick Ingram, Rafe Jones, and Alon Levy. Attracting cycles in p-adic dynamics and height bounds for postcritically finite maps. *Duke Math. J.* **163**(2014), 2325–2356.

[Be] Robert L. Benedetto. *Dynamics in one non-archimedean variable*, volume 198 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2019.

[BFN] Maxime Bergeron, Khashayar Filom, and Sam Nariman. Topological aspects of the dynamical moduli space of rational maps. *Adv. Math.* **397**(2022), Paper No. 108209, 40 pp.

[Be] François Berteloot. Lyapunov exponent of a rational map and multipliers of repelling cycles. *Riv. Math. Univ. Parma (N.S.)* **1**(2010), 263–269.

[BE] Xavier Buff and Adam Epstein. Bifurcation measure and postcritically finite rational maps. In *Complex dynamics*, pages 491–512. A K Peters, Wellesley, MA, 2009.

[CP] Dawei Chen and Miguel Prado. Counting differentials with fixed residues. *Preprint,* arXiv:2307.04221 [math.AG].

[De1] Laura DeMarco. Dynamics of rational maps: a current on the bifurcation locus. *Math. Res. Lett.* **8**(2001), 57–66.

[De2] Laura DeMarco. Dynamics of rational maps: Lyapunov exponents, bifurcations, and capacity. *Math. Ann.* **326**(2003), 43–73.

[De3]   Laura DeMarco. Dynamical moduli spaces and elliptic curves (KAWA Lecture Notes). *Ann. Fac. Sci. Toulouse Math.* **27**(2018), 389–420.

[Du]   Romain Dujardin. The supports of higher bifurcation currents. *Ann. Fac. Sci. Toulouse Math. (6)* **22**(2013), 445–464.

[DF]   Romain Dujardin and Charles Favre. Distribution of rational maps with a preperiodic critical point. *Amer. J. Math.* **130**(2008), 979–1032.

[DFG]   Romain Dujardin, Charles Favre, and Thomas Gauthier. When do two rational functions have locally biholomorphic Julia sets? *Preprint,* arXiv:2201.04116 [math.DS].

[FG]   Charles Favre and Thomas Gauthier. *The Arithmetic of Polynomial Dynamical Pairs,* volume 214 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2022.

[Fu]   Masayo Fujimura. The moduli space of rational maps and surjectivity of multiplier representation. *Comput. Methods Funct. Theory* **7**(2007), 345–360.

[Ga1]   Thomas Gauthier. Strong bifurcation loci of full Hausdorff dimension. *Ann. Sci. Éc. Norm. Supér. (4)* **45**(2012), 947–984 (2013).

[Ga2]   Thomas Gauthier. Good height functions on quasiprojective varieties: equidistribution and applications in dynamics. *Preprint,* arXiv:2105.02479v3 [math.NT].

[Ge]   Lukas Geyer. Personal communication, 2003.

[Gor]   Igors Gorbovickis. Algebraic independence of multipliers of periodic orbits in the space of rational maps of the Riemann sphere. *Mosc. Math. J.* **15** (2015), 73–87, 181–182.

[Got]   Rin Gotou. Dynamical systems of correspondences on the projective line II: degrees of multiplier maps. *Preprint,* arXiv:2309.15404 [math.DS].

[HT]   Benjamin Hutz and Michael Tepper. Multiplier spectra and the moduli space of degree 3 morphisms on $\mathbb{P}^1$. *JP J. Algebra Number Theory Appl.* **29**(2013), 189–206.

[JX1]   Zhuchao Ji and Junyi Xie. DAO for curves. *Preprint,* arXiv:2302.02583v2 [math.DS].

[JX2]   Zhuchao Ji and Junyi Xie. Local rigidity of Julia sets. *Preprint,* arXiv:2302.02562 [math.DS].

[JX3]   Zhuchao Ji and Junyi Xie. The multiplier spectrum morphism is generically injective. *Preprint,* arXiv:2309.15382 [math.DS].

[JXZ]   Zhuchao Ji, Junyi Xie, and Geng-Rui Zhang. Space spanned by characteristic exponents. *Preprint,* arXiv:2308.00289 [math.DS].

[LP]   G. Levin and F. Przytycki. When do two rational functions have the same Julia set? *Proc. Amer. Math. Soc.* **125**(1997), 2179–2190.

[Le1]   Genadi M. Levin. Symmetries on Julia sets. *Mat. Zametki* **48**(1990), 72–79, 159.

[Le2]   Alon Levy. The space of morphisms on projective space. *Acta Arith.* **146**(2011), 13–31.

[Ly]   M. Yu. Lyubich. Some typical properties of the dynamics of rational mappings. *Uspekhi Mat. Nauk* **38**(1983), 197–198.

[MSS]   R. Mañé, P. Sad, and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. Ec. Norm. Sup.* **16**(1983), 193–217.

[MS]   Niki Myrto Mavraki and Harry Schmidt. On the dynamical Bogomolov conjecture for families of split rational maps. *Preprint,* arXiv:2201.10455v3 [math.NT].

[Mc1]   Curtis T. McMullen. Families of rational maps and iterative root-finding algorithms. *Ann. of Math. (2)* **125**(1987), 467–493.

[Mc2]   Curtis T. McMullen. *Complex Dynamics and Renormalization.* Princeton University Press, Princeton, NJ, 1994.

[Mi1]   John Milnor. Geometry and dynamics of quadratic rational maps. *Experiment. Math.*
         2(1993), 37–83. With an appendix by the author and Lei Tan.
[Mi2]   John Milnor. *Dynamics in One Complex Variable*, volume 160 of *Annals of Mathematics
         Studies*. Princeton University Press, Princeton, NJ, Third edition, 2006.
[Mi3]   John Milnor. On Lattès maps. In *Dynamics on the Riemann sphere*, pages 9–43. Eur.
         Math. Soc., Zürich, 2006.
[Pa1]   Fedor Pakovich. Recomposing rational functions. *Int. Math. Res. Not. IMRN*, **7**(2007),
         1921–1935.
[Pa2]   Fedor Pakovich. On iterates of rational functions with maximal number of critical values.
         Preprint, arXiv:2107.05963v4 [math.DS].
[PRS]   Feliks Przytycki, Juan Rivera-Letelier, and Stanislav Smirnov. Equivalence and topological
         invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps.
         Invent. Math. **151**(2003), 29–63.
[Si1]   Joseph H. Silverman. Integer points, Diophantine approximation, and iteration of rational
         maps. *Duke Math. J.* **71**(1993), 793–829.
[Si2]   Joseph H. Silverman. The space of rational maps on $\mathbf{P}^1$. *Duke Math. J.* **94**(1998), 41–77.
[Su]   Toshi Sugiyama. The moduli space of polynomial maps and their fixed-point multipliers.
         Adv. Math. **322**(2017), 132–185.
[Ta]   Lei Tan. Similarity between the Mandelbrot set and Julia sets. *Comm. Math. Phys.*
         134(1990), 587–617.
[We]   Lloyd West. The moduli space of cubic rational maps. *Preprint,* arXiv:1408.3247
         [math.NT].
[YZ1]   Xinyi Yuan and Shou-Wu Zhang. The arithmetic Hodge index theorem for adelic line
         bundles. *Math. Ann.* **367**(2017), 1123–1171.
[YZ2]   Xinyi Yuan and Shouwu Zhang. Adelic line bundles over quasiprojective varieties. *Preprint,*
         arXiv:2105.13587v5 [math.NT].

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