MAT 240H1F - ALGEBRA I

Review questions

Note: Solutions to these questions will not be provided.

1. Problem Set 1: Questions 1–6, 9.
2. Problem Set 2: Questions 1–5.
3. Problem Set 3: Questions 1–5, 7, 9, 10, 11a).
4. Problem Set 4: Questions 1–3, 4, 5 6a), the other 6, 7a), 8.
5. Let x and y be nonzero elements in a field F. Using *only* the axioms in the definition of field, one at a time, prove that there exists a *unique* $z \in F$ such that $x = yz$. Write out complete statements of all axioms which are used in your proof.
6. Let $F = \{ (a_1, a_2) \mid a_1, a_2 \in \mathbb{R} \}$. If (a_1, a_2), $(b_1, b_2) \in F$, define

$$\begin{aligned} (a_1, a_2) + (b_1, b_2) &= (a_1 + b_1, a_2 + b_2) && \text{(addition in } F\text{)} \\ (a_1, a_2) \cdot (b_1, b_2) &= (a_1 b_1 + 3\, a_2 b_2, a_2 b_1 + a_1 b_2) && \text{(multiplication in } F\text{)} \end{aligned}$$

   a) Find a multiplicative inverse of the element $(\sqrt{5}, 1)$ in F.
   b) Prove that F is not a field relative to the above addition and multiplication by finding an example of a nonzero element of F which does not have a multiplicative inverse. (*Hint*: Use the fact that 3 is a square in the real numbers.)

7. Let $\mathbb{F}_7$ be the finite field containing 7 elements. That is, $\mathbb{F}_7 = \{ 0, 1, 2, 3, 4, 5, 6 \}$, with addition and multiplication modulo 7. Let $F = \{ (a_1, a_2) \mid a_1,\ a_2 \in \mathbb{F}_7 \}$. If (a_1, a_2), $(b_1, b_2) \in F$, define

$$\begin{aligned} (a_1, a_2) + (b_1, b_2) &= (a_1 + b_1 (\operatorname{mod} 7), a_2 + b_2 (\operatorname{mod} 7)) && \text{(addition in } F\text{)} \\ (a_1, a_2) \cdot (b_1, b_2) &= ((a_1 b_1 + 4\, a_2 b_2)(\operatorname{mod} 7), (a_2 b_1 + a_1 b_2)(\operatorname{mod} 7)) && \text{(multiplication in } F\text{)} \end{aligned}$$

   a) Find an additive inverse of $(1, 3)$ in F.
   b) Find a multiplicative inverse of $(2, 2)$ in F.
   c) Explain why the element $(5, 1)$ does not have a multiplicative inverse in F.

8. Let V be a vector space over a field F. Let x and y be vectors in V. Using only the axioms from the definition of vector space (one axiom at a time) prove that the vector $(-x) + (-y)$ (the sum of $-x$ and $-y$) is an additive inverse of the vector $x + y$. Give a statement of each axiom that you use in your proof.
9. Let $V = \{ p(x) = a_2 x^2 + a_1 x + a_0 \mid a_2, a_1, a_0 \in \mathbb{R} \}$ be the set of polynomials of degree less than or equal to 2, having real coefficients. Let addition of elements of V be defined by

$$(p_1 + p_2)(x) = p_1(x) + p_2(x), \quad p_1, p_2 \in V,$$

   and let scalar multiplication be defined by

$$(cp)(x) = p(cx), \quad c \in \mathbb{R},\ p \in V.$$

   Do the above addition and scalar multiplication satisfy the vector space axioms? Justify your answer.

10. Let $\mathbb{F}_5$ be the finite field containing 5 elements. That is, $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$, with addition and multiplication modulo 5. Let $V = \{(a_1, a_2) \mid a_1,\ a_2 \in \mathbb{F}_5\}$. If (a_1, a_2), $(b_1, b_2) \in V$ and $c \in \mathbb{F}_5$, let

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1 (\mathrm{mod}\, 5), a_2 + b_2 (\mathrm{mod}\, 5))$$
$$c(a_1, a_2) = \begin{cases} (0, 0), & \text{if } c = 0 \\ (ca_1 (\mathrm{mod}\, 5), c^{-1}a_2 (\mathrm{mod}\, 5)), & \text{if } c \in \mathbb{F}_5 \text{ and } c \neq 0. \end{cases}$$

Determine whether or not the set V is a vector space over $\mathbb{F}_5$ with the above addition and scalar multiplication. (Here, if $a,\ b \in \mathbb{F}_5$, $a + b(\mathrm{mod}\, 5)$ is the sum of a and b in $\mathbb{F}_5$, and $ab(\mathrm{mod}\, 5)$ is the product of a and b in $\mathbb{F}_5$.) If V is a vector space, show that each of the vector space axioms holds. If V is not a vector space, demonstrate how one of the axioms fails to hold.

11. Determine whether the set $V = \mathbb{R}^2$ together with the vector addition and scalar multiplication below is a vector space over $\mathbb{R}$. If so, prove that all the axioms in the definition of vector space are satisfied. If not, show by example that at least one axiom does not hold.

    vector addition: $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2)$
    scalar multiplication: $c(x_1, x_2) = (cx_1 + c - 1, cx_2)$

12. In each case below, determine whether or not the subset W of the vector space V is a subspace of V. If W is a subspace of V, prove it. If W is not a subspace of V, demonstrate how one of the properties of subspace fails to hold.

    a) Let $V = P(\mathbb{R})$ (the vector space of polynomials in one variable, with coefficients in $\mathbb{R}$). Let $W = \{f(x) \in P(\mathbb{R}) \mid f(0) \geq 0\}$.

    b) Let $V = M_{2x2}(\mathbb{Q})$ (the vector space of 2×2 matrices with rational entries). Let

    $$W = \left\{ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_{2\times 2}(\mathbb{Q}) \;\mid\; A_{22} = 3\, A_{12} A_{21} \right\}.$$

    c) Let $V = \mathbb{C}^3$ and $W = \{(z_1, z_2, z_3) \in V \mid z_1 - iz_2 = 0\}$.

    d) Let $V = \mathbb{C}^3$ and $W = \{(z_1, z_2, z_3) \in V \mid \bar{z}_1 - iz_2 = 0\}$.

    e) Let $V = \mathbb{R}^3$ and

    $$W = \{x = (x_1, x_2, x_3) \in V \mid x_1 - x_2 \text{ is a rational number }\}$$

    f) Let n be an integer such that $n \geq 2$. Let $V = P_n(\mathbb{C})$ be the vector space of polynomials of degree at most n, with complex coefficients. Define

    $$W = \{f \in V \mid f(i) + 2\, f(-i) = 0\}.$$

    g) Let V be the vector space of functions from the real numbers $\mathbb{R}$ to the vector space $\mathbb{R}^3$. Vector addition in V and and scalar multiplication are defined as follows: If f and g are in V, x is a real number, and $f(x) = (a_1, a_2, a_3)$, $g(x) = (b_1, b_2, b_3) \in \mathbb{R}^3$, then

    $$(f + g)(x) = f(x) + g(x) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$
    $$(cf)(x) = (ca_1, ca_2, ca_3), \qquad \text{for } c \in \mathbb{R}$$

    Let $W = \{f \in V \mid f(\sqrt{2}) \in \mathrm{span}\{(1, 0, 0), (0, 0, 1)\}\}$.

    h) Let V be the vector space of functions from $\mathbb{C}$ to $\mathbb{C}$ (where $\mathbb{C}$ is the complex numbers), with vector addition given by addition of functions:$(f + g)(z) = f(z) + g(z)$ for all $z \in \mathbb{C}$, and scalar multiplication defined by $(c\,f)(z) = c \cdot f(z)$ for $c \in \mathbb{C}$ and $z \in \mathbb{C}$. Let $W = \{f \in V \mid f(iz) = -f(z) \text{ for all } z \in \mathbb{C}\}$.

13. In each case below, determine whether or not the subset S of the vector space V is linearly independent. In addition, compute the dimension of the subspace span(S). (Justify your answers fully).
    a) Let V be the space of functions from $\mathbb{R}$ to $\mathbb{R}$, and let $S = \{ f(t) = e^{rt},\ g(t) = e^{st} \}$, where r and s are distinct real numbers.
    b) Let $V = P(\mathbb{R})$. Let $S = \{ x^2 - 1,\ x^2 + 1, x^2 - 2, x^2 + 2 \}$.
    c) Let $V = \mathbb{R}^3$ and let $S = \{ x = (x_1, x_2, x_3) \in V \mid x_1 - x_2 = 0 \}$.
    d) Let $V = M_{2\times 2}(\mathbb{R})$, and let S be the set given in 2g) or 2h) of section 1.5 of the text.
    e) Let $V = P_{2n}(F)$ ($n \geq 1$, F a field) and let $c_1, c_2, \ldots, c_n \in F$ be nonzero scalars. (That is, $c_j \neq 0$, $1 \leq j \leq n$). Let $S = \{ c_1, c_2\, x^2, c_3\, x^6, \ldots, c_j\, x^{2j}, \ldots, c_n\, x^{2n} \}$.
    f) Let V be a vector space over a field F and let $S = \{ -y + z,\ x + y,\ c\, z \}$, where x, y and z are distinct vectors in V such that $S' = \{ x,\ y,\ z \}$ is a linearly independent subset of V, and c is a nonzero element of F.
    g) Let V be a vector space over the field $\mathbb{R}$ and let $S = \{ x + z,\ x + y,\ 3\,x - y + 4\,z \}$, where x, y and z are distinct vectors in V such that $S' = \{ x,\ y,\ z \}$ is a linearly independent subset of V.

14. Find a basis for each of the indicated vector spaces W.
    a) Let W be the subspace of $\mathbb{C}^4$ consisting of those vectors $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ which satisfy
    $$x_1 - ix_2 + x_4 = 0$$
    $$x_1 + 2x_4 = 0.$$
    b) Let W be the subspace of vectors $f(x) \in V = P_3(\mathbb{C})$ such that $2\,f(0) = f(1)$.
    c) Let W be the subspace of matrices $A = (a_{ij}) \in M_{3\times 3}(F)$ (F a field) which satisfy $a_{ij} = a_{ji}$ for $1 \leq i,\ j \leq 3$.
    d) Let W be the subspace of polynomials $f(x) \in P_n(\mathbb{R})$ such that $f(x) = -f(-x)$ for all x.

15. a) Find a basis of $P_2(\mathbb{Q})$ having the property that every polynomial $f(x)$ that belongs to the basis satisfies $f(1) \neq f(-1)$. (Be sure to explain why the set that you define is a basis).
    b) Let S be a basis for $P_2(\mathbb{Q})$. Prove that at least one $f(x)$ in the basis must satisfy $f(1) \neq f(-1)$.

16. Let $P_n(\mathbb{R})$ be the vector space of real polynomials of degree less than or equal to n, with the usual vector addition and scalar multiplication.
    a) Find a basis of $P_2(\mathbb{R})$ consisting of polynomials of equal degree.
    b) Prove that a basis of $P_n(\mathbb{R})$ must contain at least one polynomial of degree n.
    c) Prove that $\{ 1, x + 1, (x + 1)^2, \ldots, (x + 1)^n \}$ is a basis of $P_n(\mathbb{R})$.

17. Let $n \geq 2$ and let V be an n-dimensional vector space over a field F. Let j be an integer such that $1 \leq j \leq n - 1$. Prove that there exists at least one subspace W of V such that $\dim(W) = j$.

18. Let V be a finite-dimensional vector space over a field F. Suppose that W_1 and W_2 are nonzero subspaces of V such that $W_1 \cap W_2 = \{\mathbf{0}\}$. Prove that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$. (Here, $W_1 + W_2 = \{ x + y \mid x \in W_1,\ y \in W_2 \}$ is the sum of the subspaces W_1 and W_2. Recall that, as shown on one of the problem sets, $W_1 + W_2$ is a subspace of V.)

19. Questions from text: §1.2, 12–19, 21; §1.3, 8–13, 18, 20, 22; §1.4, 6, 12–14; §1.5, 7–10, 13, 14; §1.6, 11–16, 23, 26.