

MAT 240, Fall 2008
Solutions to Problem Set 8

1. (a) Let A be a lower triangular matrix. We shall develop the determinant of A by the first column:

$$\begin{aligned} \det \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix} &= a_{11} \det \begin{pmatrix} a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & \cdots & a_{3n} \\ 0 & 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix} + 0 \\ &= a_{11}a_{22} \det \begin{pmatrix} a_{33} & a_{34} & \cdots & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & \cdots & a_{4n} \\ 0 & 0 & a_{55} & \cdots & a_{5n} \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix} + 0 \\ &= \cdots = \\ &= a_{11}a_{22} \cdots a_{nn}. \end{aligned}$$

Similar calculations can be done for an upper triangular matrix.

- (b) Let the matrix A from part a) be such that $A = [T]_{\beta}$. Then, to calculate the eigenvalues of T , we need to find zeros of characteristic polynomial $\det(A - \lambda I_n)$.

$$\det(A - \lambda I_n) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} - \lambda & \cdots & \cdots & a_{2n} \\ 0 & 0 & a_{33} - \lambda & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & a_{nn} - \lambda \end{pmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$$

Hence, eigenvalues of T are $a_{11}, a_{22}, \dots, a_{nn}$.

2. Here, we work with a standard basis of \mathbb{R}^6 in which T has a matrix $[T]$. The characteristic polynomial of a linear operator T is $f(t) = \det([T] - tI_6) = (t - 4)(t + 1)^3(t - 2)^2$. This means that $\det([T] - tI_6) = 0$ if and only if $t = -1, 2$, or 4 ; otherwise, $\det([T] - tI_6) \neq 0$. A linear transformation is invertible $T^3 + 2T^2 - 3T$ if and only if $\det([T^3 + 2T^2 - 3T]) \neq 0$. So,

$$\begin{aligned} \det([T^3 + 2T^2 - 3T]) &= \det([T]([T]^2 + 2[T] - 3I_6)) \\ &= \det([T]) \det([T] + 3I_6)([T] - I_6) \\ &= \det([T]) \det([T] + 3I_6) \det([T] - I_6) \\ &\neq 0, \end{aligned}$$

since here we have product of three non-zero factors of form $\det([T] - tI_6)$ for $t = 0, -3, 1$.

3. (a) In basis $\beta = \{x^3, x^2, x, 1\}$, we have

$$[T] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ -2 & -1 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} \det([T] - \lambda I_4) &= \det \begin{pmatrix} 1 - \lambda & 0 & 0 & 0 \\ 1 & -3 - \lambda & 1 & 1 \\ -2 & -1 & -1 - \lambda & -2 \\ 0 & 0 & 0 & 1 - \lambda \end{pmatrix} \\ &= (1 - \lambda) \det \begin{pmatrix} -3 - \lambda & 1 & 1 \\ -1 & -1 - \lambda & -2 \\ 0 & 0 & 1 - \lambda \end{pmatrix} \\ &= (1 - \lambda)^2 \det \begin{pmatrix} -3 - \lambda & 1 \\ -1 & -1 - \lambda \end{pmatrix} \\ &= (1 - \lambda)^2 (\lambda + 2)^2 \end{aligned}$$

The $\lambda_1 = 1$ and $\lambda_2 = -2$ are eigenvalues of T , both with multiplicity 2.

(b) Rank of $T - \lambda I_V$ equals rank of matrix $[T - \lambda I_V]$, which is the number of linearly independent row or columns of the matrix. We compute it by row and column reducing the matrix, till we can say how many rows or columns are linearly independent.

$$[T - \lambda_1 I_V] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 1 \\ -2 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow R_3 := R_3 + 2R_2 \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 1 \\ 0 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are two linearly independent rows, $\text{rank}(T - I_V) = 2$.

Similarly, we determine that

$$[T + 2I_V] = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ -2 & -1 & 1 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

has rank 3.

(c) Eigenvalue $\lambda_1 = 1$ has multiplicity $2 = \text{nullity}(T - I_V) = \dim(V) - \text{rank}(T - I_V) = 4 - 2$. However, eigenvalue $\lambda_2 = -2$ has multiplicity $2 \neq \text{nullity}(T + 2I_V) = 4 - 3 = 1$. By test for diagonalization, T is not diagonalizable.

In other words, if T is diagonalizable, we can find a basis of $P_3(\mathbb{R})$ consisting of eigenvectors of T . Eigenvectors with eigenvalue λ_i , $i = 1, 2$, satisfy $T(v) = \lambda_i v$, i.e. $(T - \lambda_i I_V)(v) = 0$. Therefore, linearly independent eigenvectors with eigenvalue λ_i form a basis of $N(T - \lambda_i)$. However, we know that $\text{nullity}(T - I_V) = \dim(V) - \text{rank}(T - I_V) = 4 - 2 = 2$, so there are two linearly independent eigenvectors with the eigenvalue 1; and $\text{nullity}(T + 2I_V) = 4 - 3 = 1$, so there is only one linearly independent vector with eigenvalue -2 . Since there are overall only three linearly independent eigenvectors, we cannot form a basis of $P_3(\mathbb{R})$ in which T would be diagonal. Therefore, T is not diagonalizable.

- (d) As we mentioned, $E_1 = N(T - I_V) = \{\vec{x} = (a, b, c, d) \mid [T - I_V]\vec{x} = \vec{0}\}$. Hence, we need to find solutions of system,

$$\begin{aligned} 0 &= 0 \\ a - 4b + c + d &= 0 \\ -2a - b - 2c - 2d &= 0 \\ 0 &= 0 \end{aligned}$$

So,

$$E_1 = \left\{ \left(\begin{pmatrix} -c-d \\ 0 \\ c \\ d \end{pmatrix} : c, d \in \mathbb{R} \right) = \left\{ c \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} : c, d \in \mathbb{R} \right\} \right\}.$$

Hence, a basis of E_1 is $\{-x^3 + x, -x^3 + 1\}$. We can check that $T(-x^3 + x) = -x^3 + x$ and $T(-x^3 + 1) = -x^3 + 1$.

4. (a) The characteristic polynomial of T is $f(\lambda) = \det([T] - \lambda I_4) = (\lambda - 1)^2(\lambda - 2i)^2$. Therefore, the eigenvalues of T are $\lambda_1 = 1$ and $\lambda_2 = 2i$.
 (b) We need to determine rank of

$$[T - I_V] = \begin{pmatrix} 2i & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 - 2i & 2 - 2i & -2 + 2i \end{pmatrix} \text{ and } [T - 2iI_V] = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 - 2i & 0 & 0 \\ 0 & 0 & 1 - 2i & 0 \\ 1 & 2 - 2i & 2 - 2i & -1 \end{pmatrix}.$$

After some row and column reductions, we deduce that $\text{rank}(T - I_V) = \text{rank}([T - I_V]) = 2$ and $\text{rank}(T - 2iI_V) = \text{rank}([T - 2iI_V]) = 3$.

- (c) Multiplicity of eigenvalue $\lambda_1 = 1$ is 2 and $\text{nullity}(T - I_V) = 4 - 2 = 2$. But, eigenvalue $\lambda_2 = 2i$ has multiplicity $2 \neq \text{nullity}(T - 2iI_V) = 4 - 3 = 1$. Therefore, T is not diagonalizable.
 5. (a) We have that $T^4(f(x)) = T^3(f(ix)) = T^2(f(-x)) = T(f(-ix)) = f(x)$, for all $f \in P(\mathbb{C})$. Hence, $T^4 = I_V$.
 (b) If λ is an eigenvalue of T , then for some function $f \in P(\mathbb{C})$, $f \neq 0$, $T(f) = \lambda f$. So, $f = T^4(f) = \lambda^4 f$, implying that $(1 - \lambda^4)f = 0$. Since $f \neq 0$, $\lambda^4 = 1$. Therefore, the eigenvalues of T are 1, -1, i , and $-i$.
 (c) Basis of $N(T - I_V)$ is $\{x^{4k} \mid k = 0, 1, 2, \dots\}$. Basis of $N(T - iI_V)$ is $\{x^{4k+1} \mid k = 0, 1, 2, \dots\}$. Basis of $N(T + I_V)$ is $\{x^{4k+2} \mid k = 0, 1, 2, \dots\}$. Finally, basis of $N(T + iI_V)$ is $\{x^{4k+3} \mid k = 0, 1, 2, \dots\}$.
 6. (a) If $\text{nullity}(T^2 - T) > 0$, there there exists $v \in V$, $v \neq 0$, such that $(T^2 - T)(v) = 0$. Therefore, $T \circ (T - I_V)(v) = 0$. If $(T - I_V)(v) = 0$, then $T(v) = v$, so v is an eigenvector with eigenvalue 1. Otherwise, $(T - I_V)(v) = w$, $w \neq 0$. Then, $T \circ (T - I_V)(v) = T(w) = 0$, so w is an eigenvector with eigenvalue 0. Hence, at least one of 1 and 0 is an eigenvalue of T .

- (b) Let β be a basis of $N(T)$, and let γ be a basis of $N(T - I_V)$. By Theorem 5.8 (which holds since V is finite dimensional), $\beta \cup \gamma$ is a linearly independent set which consists of $\text{nullity}(T) + \text{nullity}(T - I_V)$ vectors. Moreover, we claim that $\beta \cup \gamma \subset N(T^2 - T)$. Indeed, if $x \in \beta$, then $T(x) = 0$, so $(T^2 - T)(x) = T^2(x) - T(x) = 0 - 0 = 0$. Also, if $x \in \gamma$, then $(T - I_V)(x) = 0$, i.e. $T(x) = I_V(x) = x$. Therefore, $(T^2 - T)(x) = T^2(x) - T(x) = T(x) - x = x - x = 0$. Hence, $\text{nullity}(T^2 - T) \geq |\beta \cup \gamma| = \text{nullity}(T) + \text{nullity}(T - I_V)$.
- (c) If T is diagonalizable, there exists a basis β of V in which $[T]_\beta$ is a diagonal matrix. Let

$$[T]_\beta = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a_n \end{pmatrix}.$$

Then, $\text{nullity}(T) = n - \text{rank}(T)$. We know $\text{rank}(T)$ equals number of linearly independent rows, that is number of diagonal elements a_i , $i = 1, 2, \dots, n$, such that $a_i \neq 0$. That is $\text{nullity}(T)$ equals number of a_i 's such that $a_i = 0$. Since

$$[T - I_V]_\beta = \begin{pmatrix} a_1 - 1 & 0 & \cdots & 0 \\ 0 & a_2 - 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a_n - 1 \end{pmatrix},$$

$\text{nullity}(T - I_V)$ equals number of a_i 's such that $a_i - 1 = 0$, i.e. $a_i = 1$. Now,

$$[T^2 - T]_\beta = [T]_\beta^2 - [T]_\beta = \begin{pmatrix} a_1^2 - a_1 & 0 & \cdots & 0 \\ 0 & a_2^2 - a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a_n^2 - a_n \end{pmatrix},$$

so $\text{nullity}(T^2 - T)$ equals number of a_i 's for which $a_i^2 - a_i = a_i(a_i - 1) = 0$, that is, number of a_i 's such that $a_i = 0$ or $a_i = 1$. Hence, $\text{nullity}(T^2 - T) = \text{nullity}(T) + \text{nullity}(T - I_V)$.

- (d) Let β be a basis of $N(T)$ and γ be a basis of $N(T - I_V)$. We have proved that $\beta \cup \gamma$ is a linearly independent set in $N(T^2 - T)$. In order to prove it is a basis of $N(T^2 - T)$, we need to show that it generates $N(T^2 - T)$. That is, if $x \in N(T^2 - T)$, we need to show that x can be written as a linear combination of vectors in $\beta \cup \gamma$. Let $x = (x - T(x)) + T(x)$. Then, $(T - I_V)(T(x)) = (T^2 - T)(x) = 0$, so $T(x) \in \text{span}(\gamma)$. Also, $T(x - T(x)) = T \circ (I_V - T)(x) = -(T^2 - T)(x) = 0$, so $x - T(x) \in \text{span}(\beta)$. Therefore, $x \in \text{span}(\beta \cup \gamma)$.

7. \Rightarrow : Assume that T is diagonalizable. Then in some basis β ,

$$[T]_\beta = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Hence, $\det([T]_\beta) = a_{11}a_{22}\cdots a_{nn} \neq 0$ since T is invertible. Therefore, $a_{ii} \neq 0$ for all $i =$

$1, 2, \dots, n$. We can compute T^{-1} ,

$$[T^{-1}]_{\beta} = \begin{pmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{pmatrix}.$$

So, T^{-1} is diagonal.

\Leftarrow : This direction is the same as above, just start from the assumption that T^{-1} is diagonalizable.

8. \Rightarrow : Assume that T is diagonalizable, that is $[T]_{\beta} = D$ is a diagonal matrix in a basis β . Consider the new basis γ defined by the change of basis matrix $[U]_{\beta} = [I_n]_{\gamma}^{\beta}$ (that is, if $\gamma = \{y_1, y_2, \dots, y_n\}$, the coordinates of y_i in basis β are given by i^{th} column of $U = [c_1 \ c_2 \ \cdots \ c_n]$, i.e. $[y_i]_{\beta} = c_i$). Then in basis γ , $[UTU^{-1}]_{\gamma} = [I_n]_{\beta}^{\gamma} [UTU^{-1}]_{\beta} [I_n]_{\gamma}^{\beta} = [U]_{\beta}^{-1} [U]_{\beta} [T]_{\beta} [U]_{\beta}^{-1} [U]_{\beta} = [T]_{\beta} = D$.

\Leftarrow : We can do the same in this direction. Given a basis β in which UTU^{-1} is diagonalizable, consider change of matrix given by $[I_n]_{\gamma}^{\beta} = [U]_{\beta}$.

9. If A and B are similar, there exists an invertible matrix C such that $A = C^{-1}BC$. We can define a linear operator W given by $[W]_{\beta} = C$. Then this problem reduced to proving that T is diagonalizable if and only if $W^{-1}UW$ is diagonalizable. We proved this in the previous problem.

10. Let β be a basis in which T is diagonalizable. Then,

$$[T]_{\beta} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

In complex numbers, $x^2 - c = 0$ has two solutions for any $c \in \mathbb{C}$. Hence, we can always find $b_{ii}^2 = a_{ii}$, $i = 1, 2, \dots, n$. Let $U \in \mathcal{L}(\mathcal{V})$ be defined by

$$[U]_{\beta} = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix}.$$

Then, $U^2 = T$ since $[U^2]_{\beta} = [U]_{\beta}^2 = [T]_{\beta}$.

11. Problems from section §5.1 :

#8.

(a) We will prove this by counter position.

\Rightarrow : Assume that zero is an eigenvalue of T . Then there exists $v \in V$, $v \neq 0$, such that $T(v) = 0 \cdot v = 0$. Hence, both 0 and v are mapped by T to 0 . Hence, T is not injective, and so not invertible.

\Leftarrow : If T is not invertible, then T is not one-to-one. Otherwise, if T is injective, by dimension theorem, $\text{rank}(T) = \text{dim}(V) - \text{nullity}(T) = \text{dim}(V)$, so T is onto and hence T is invertible. So, if T is not one-to-one, then there exist $v_1, v_2 \in V$ such that $v_1 \neq v_2$ and $T(v_1) = T(v_2)$. So $T(v_1 - v_2) = 0$, which means that $v_1 - v_2 \neq 0$ is an eigenvector with eigenvalue 0.

- (b) In previous part, we have proved that eigenvalue $\lambda \neq 0$, since T is invertible. This means that λ^{-1} exists. For some $v \in V$, $v \neq 0$, such that $T(v) = \lambda v$. So $v = T^{-1}(T(v)) = T^{-1}(\lambda v) = \lambda T^{-1}(v)$. Hence, $T^{-1}(v) = \lambda^{-1}v$, i.e. λ^{-1} is an eigenvalue of T^{-1} . We prove the opposite direction similarly.

#15.

- (a) We prove this by induction. Base step: $T^2(x) = T(T(x)) = T(\lambda x) = \lambda T(x) = \lambda^2 x$.
 Induction hypothesis: Assume that for any $m \in \mathbb{N}$, $T^m(x) = \lambda^m x$.
 Induction step: Then $T^{m+1}(x) = T(T^m(x)) = T(\lambda^m x) = \lambda^m T(x) = \lambda^{m+1} x$.

#17.

- (a) Assume that λ is an eigenvalue of T . Then for some matrix $A \neq 0$, $A^t = T(A) = \lambda A$. That is,

$$A^t = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \\ \lambda a_{n1} & \lambda a_{n2} & \cdots & \lambda a_{nn} \end{pmatrix} = \lambda A.$$

Since $A \neq 0$, there exists $a_{ij} \neq 0$, and from above we see that $a_{ji} = \lambda a_{ij}$ and $a_{ij} = \lambda a_{ji}$. That is $a_{ij} = \lambda^2 a_{ij}$. Since $a_{ij} \neq 0$, $\lambda^2 = 1$, so $\lambda = \pm 1$.

- (b) Let $\lambda_1 = 1$. Then we need to find a basis for set of all matrices A such that $A = A^t$. These are symmetric matrices, which look like

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} = \sum_{i \leq j} a_{ij} A_{ij},$$

where $A_{ij} = (c_{kl})$ and $c_{kl} = 1$ if $k = i$ and $l = j$, or $k = j$ and $l = i$, otherwise $c_{kl} = 0$. Note that $\beta = \{A_{ij} \mid i \leq j, i = 1, 2, \dots, n, j = 1, 2, \dots, n\}$ is linearly independent and dimension of this space is $n + \frac{(n-1)n}{2}$.

If $\lambda_2 = -1$, then we need to find a basis of set of all matrices A such that $A^t = -A$. These matrices are

$$A = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{pmatrix} = \sum_{i < j} a_{ij} B_{ij},$$

where $B_{ij} = (c_{kl})$, $c_{ij} = 1$, $c_{ji} = -1$, otherwise $c_{kl} = 0$. The set $\gamma = \{B_{ij} \mid i < j, i = 1, 2, \dots, n, j = 1, 2, \dots, n\}$ is linearly independent generating set, and dimension of this space is $\frac{(n-1)n}{2}$.

(c) – (d) By Theorem 5.8, we know that $\beta \cup \gamma$ is a linearly independent set having $(n + \frac{(n-1)n}{2}) + \frac{(n-1)n}{2} = n^2$ vectors. Hence, it is a basis of $M_{n \times n}(\mathbb{R})$, and in this basis T is diagonal.

#22.

(a) We have shown that $T^m(x) = \lambda^m x$ in problem #15. Let $g(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$. Then, $g(T)$ is a linear map and

$$\begin{aligned} g(T)(x) &= a_n T^n(x) + a_{n-1} T^{n-1}(x) + \cdots + a_1 T(x) + a_0 I_v(x) \\ &= a_n \lambda^n x + a_{n-1} \lambda^{n-1} x + \cdots + a_1 \lambda x + a_0 x \\ &= g(\lambda)x. \end{aligned}$$

Problems from section §5.2:

#8.

Since λ_2 is an eigenvalue of A , there exists at least one vector $x \in M_{n \times 1}$ such that $A \cdot x = \lambda_2 x$. Hence, $\dim(E_{\lambda_2}) \geq 1$. Since $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) \leq n$ in general, and here $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) \geq n$, we conclude that $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) = n$ and hence A is diagonalizable.

#12.

(a) Let $E(T)_\lambda$ be the eigenspace of T corresponding to eigenvalue λ , and let $E(T^{-1})_{\lambda^{-1}}$ be the eigenspace of T^{-1} corresponding to λ^{-1} . We will prove that $E(T)_\lambda \subset E(T^{-1})_{\lambda^{-1}}$ and $E(T^{-1})_{\lambda^{-1}} \subset E(T)_\lambda$. Let $x \in E(T)_\lambda$. Then $T(x) = \lambda x$, that is $x = T^{-1}(T(x)) = \lambda T^{-1}(x)$, so $T^{-1}(x) = \lambda^{-1}x$, so $x \in E(T^{-1})_{\lambda^{-1}}$. Similarly we prove another inclusion. Hence, $E(T)_\lambda = E(T^{-1})_{\lambda^{-1}}$.

(b) Look at problem 7. in these solutions.