MAT 240F - Problem Set 8

Not to be handed in.

Remark: With the exception of questions 5c) and 6d), the questions in this problem set can be used as review questions for the final exam.

1. A matrix $A = (a_{jk}) \in M_{n\times n}(F)$ is upper triangular if $a_{jk} = 0$ whenever $j > k$. (That is, all of the matrix entries below the diagonal are equal to 0.) The matrix A is lower triangular if $a_{jk} = 0$ whenever $j < k$.
   a) Prove that the determinant of an upper or lower triangular matrix A is equal to the product $a_{11}a_{22}\cdots a_{nn}$ of the diagonal entries of A.
   b) Suppose that $T \in \mathcal{L}(V)$, $\dim(V) = n$, and there exists an ordered basis β for V such that $A = [T]_\beta$ is an upper triangular or lower triangular matrix. Prove that the eigenvalues of T are the diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ of A.
2. Let $T \in \mathcal{L}(\mathbb{R}^6)$ be a linear transformation whose characteristic polynomial is equal to $f(t) = (t-4)(t+1)^3(t-2)^2$. Show that $T^3 + 2T^2 - 3T$ is invertible.
3. Let $T \in \mathcal{L}(P_3(\mathbb{R}))$ be defined by

$$T(ax^3 + bx^2 + cx + d) = ax^3 + (a - 3b + c + d)x^2 + (-2a - b - c - 2d)x + d, \qquad a,\, b,\, c,\, d \in \mathbb{R}.$$

   a) Find the characteristic polynomial and all eigenvalues of T.
   b) Compute $\mathrm{rank}(T - \lambda I_V)$ for each eigenvalue λ of T. (Here, I_V is the identity transformation: $I_V(f(x)) = f(x)$ for all $f(x) \in V = P_3(\mathbb{R})$.)
   c) Determine whether T is diagonalizable. Justify your answer.
   d) Find a basis for the eigenspace E_1 of T corresponding to the eigenvalue 1.
4. Let $T \in \mathcal{L}(M_{2\times 2}(\mathbb{C}))$ be the linear transformation whose matrix $[T]_\beta$ relative to the ordered basis

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \right\}$$

   is equal to

$$[T]_\beta = \begin{pmatrix} 1+2i & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2-2i & 2-2i & -1+2i \end{pmatrix}.$$

   a) Find the characteristic polynomial and all eigenvalues of T.
   b) Compute $\mathrm{rank}(T - \lambda I_V)$ for each eigenvalue λ of T.
   c) Determine whether T is diagonalizable.
5. Let $V = P(\mathbb{C})$. Define $T \in \mathcal{L}(V)$ by $T(f(x)) = f(ix)$, $f(x) \in V$.
   a) Prove that $T^4 = I_V$.
   b) Find all eigenvalues of T. (Be sure to explain why $\mathrm{nullity}(T - \lambda I_V) > 0$ for each $\lambda \in \mathbb{C}$ that you claim is an eigenvalue of T.)
   c) Find a basis for $N(T - \lambda I_V)$ for each eigenvalue λ of T.

6. Suppose that $T \in \mathcal{L}(V)$.

   a) Prove that if $\text{nullity}(T^2 - T) > 0$, then at least one of 0 and 1 is an eigenvalue of T. (Do not assume that V is finite-dimensional.)

   For parts b)–d), assume that V is finite-dimensional.

   b) Prove that $\text{nullity}(T^2 - T) \geq \text{nullity}(T) + \text{nullity}(T - I_V)$. (*Hint*: Theorem 5.8 is useful here.)

   c) For this part, assume that T is diagonalizable. Prove that $\text{nullity}(T^2 - T) = \text{nullity}(T) + \text{nullity}(T - I_V)$.

   d) Prove that $\text{nullity}(T^2 - T) = \text{nullity}(T) + \text{nullity}(T - I_V)$ without assuming that T is diagonalizable. (*Hint*: Let β and γ be bases for $N(T)$ and $N(T - I_V)$, respectively. To prove the desired equality, it suffices to show that $\beta \cup \gamma$ is a basis for $N(T^2 - T)$. The following may be useful: If $x \in V$, then $x = (x - T(x)) + T(x)$.)

7. Let $T \in \mathcal{L}(V)$, where V is a finite-dimensional vector space. Assume that T is invertible. Prove that T is diagonalizable if and only if T^{-1} is diagonalizable.

8. Let $T, U \in \mathcal{L}(V)$, where V is a finite-dimensional vector space. Assume that U is invertible. Prove that T is diagonalizable. if and only if $UTU^{-1} = U \circ T \circ U^{-1}$ is diagonalizable.

9. Let $U, T \in \mathcal{L}(V)$, where V is a finite-dimensional vector space. Let β be an ordered basis for V, $A = [T]_\beta$ and $B = [U]_\beta$. Assume that A and B are similar matrices. Prove that T is diagonalizable if and only if U is diagonalizable.

10. Let $T \in \mathcal{L}(V)$, where V is a finite-dimensional vector space over $\mathbb{C}$. Assume that T is diagonalizable. Prove that there exists a diagonalizable $U \in \mathcal{L}(V)$ such that $U^2 = T$.

11. Review problems from text: §5.1 - # 8, 15, 17, 22 a); §5.2 - # 8, 12.