# MAT240 Problem Set 7 Solutions

November 29, 2008

**Note:** There is often more than one solution to a given problem, and so do not interpret these solutions as the only possible set. Also, only partial solutions are given for some problems when appropriate. Finally, if you see any errors, you should let your TA know.

1. Let $\{v_1, ..., v_n\}$ be a basis for the null space of $T$. Since $U_1$ is an isomorphism, there exist $w_1, ..., w_n \in W_1$ such that $U_1(w_i) = v_i$ for each $i$, and the set of $w_i$ form a linearly independent set. Since $U_1$ is an isomorphism, its null space is just $\{0\}$, and so the null space of $TU_1$ is spanned by $\{w_1, ..., w_n\}$. Since the null space of $U_2$ is $\{0\}$, then the null space of $U_2TU_1$ is spanned by $\{w_1, ..., w_n\}$ as well. Hence, the nullities of $T$ and $U_2TU_1$ agree.

2. (a) This is just question 1 with $V_1 = V_2 = W_1 = W_2 = F^n$. Fix a basis of $F^n$, and let $A$ be the matrix representation of $T$ in this basis, $U_1$ and $U_2$ be $C$ and $C^{-1}$, respectively, and $B$ be the matrix representation of $U_2TU_1$ (that is, $B = C^{-1}AC$). Then, $A$ and $B$ have the same nullities, and hence by the dimension theorem, the same ranks.

   (b) The composition of invertible matrices is invertible. If $A$ is invertible, and $B = C^{-1}AC$ for some invertible $C$, then $B$ is such a composition. Similarly, $A = CBC^{-1}$, and so is invertible if $B$ is.

   (c) If $A$ is similar to $B$, then for some invertible $C$, $A = CBC^{-1}$. Hence, $A^{-1} = (CBC^{-1})^{-1} = CB^{-1}C^{-1}$, and so $A^{-1}$ and $B^{-1}$ are similar. For the converse, replace $A$ and $B$ in the argument with $A^{-1}$ and $B^{-1}$, respectively.

   (d) If $A$ is similar to $B$, then for some invertible $C$, $A = CBC^{-1}$. Hence,
$$\begin{aligned} A^m &= (CBC^{-1})^m \\ &= CBC^{-1}CBC^{-1}...CBC^{-1} && (m \text{ times}) \\ &= CB^mC^{-1}. \end{aligned}$$

   (e) From part (d), we know that if $A = CBC^{-1}$, then, $A^3 = CB^3C^{-1}$, and so $B^3 = C^{-1}A^3C$. Thus, $B^3 = C^{-1}(-A)C = -B$.

3. (a) $T^{-1}(a + bx + cx^2) = a + (\frac{a(1+i)}{2} - \frac{b(i+1)}{2})x + (\frac{a}{2} - \frac{b(i+1)}{2} - \frac{c}{2})x^2$

   (b) $T^{-1}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = -bx^3 - cx^2 - x + a + d$ To see this, identify both $P_3(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$ with $\mathbb{R}^4$. Then $T$ has the following representation in the standard bases:
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$
   Calculate its inverse in the usual way to obtain
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Act this on a vector $(a, b, c, d)$, and we get the vector corresponding to the polynomial above.

(c) $T^{-1}(a, b, c) = (a + \frac{b}{2} - \frac{3c}{2}, -a + \frac{b}{2} + \frac{3c}{2}, \frac{-b}{2} + \frac{c}{2})$

4. (a) $\text{rank}(T) = 4$

   (b) $\text{rank}(T) = 2$

   (c) $\text{rank}(T) = 4$. The matrix representation of $T$ with respect to the standard bases is

$$\begin{bmatrix} 4 & -1 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

Reduce this matrix to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. Since $A$ and $B$ are invertible, they have full rank, and so does their reduced row echelon counterparts; that is, they can be reduced via elementary matrices to the identity matrix. Let $E_i$ and $E'_j$ ($i = 1, ..., k$ and $j = 1, ..., l$) be elementary matrices such that

$$E_k...E_1 A = I_n = E'_l...E'_1 B.$$

Since elementary matrices are invertible, and their inverses are also elementary, we have the following chain of elementary matrices (and hence elementary row operations) taking $A$ to $B$:

$$(E'_1)^{-1}...(E'_l)^{-1} E_k...E_1 A = B.$$

6. Since $A$ and $B$ have the same rank, they can both be reduced to the same matrix $D$, where the first $r = \text{rank}(A)$ entries on the main diagonal are 1, and the rest of the entries are 0. In particular, there exist elementary matrices $E_1, ..., E_k \in M_{m\times m}(F)$ and $E'_1, ..., E'_l \in M_{n\times n}(F)$ such that

$$E_k...E_1 A = D = BE'_1...E'_l.$$

Note that the elementary matrices acting on $A$ correspond to elementary row operations, and those on $B$ to elementary column operations. Since elementary matrices are invertible, so are their compositions, and so letting $P = E_k...E_1$ and $Q = (E'_1...E'_l)^{-1}$, we have $PAQ = B$.

7. (a) Let $x \in R(T + U)$. Then there is some $y \in V$ such that $x = T(y) + U(y)$. Since $T(y) \in R(T)$ and $U(y) \in R(U)$, $x = T(y) + U(y) \in R(T) + R(U)$.

   (b) Note that if $X$ and $Y$ are subspaces of $W$, then $\dim(X + Y) = \dim(X) + \dim(Y) - \dim(X \cap Y)$. Thus, from part (a) we have $\dim(R(T + U)) \leq \dim(R(T) + R(U))$ and from the note we have $\dim(R(T) + R(U)) \leq \dim(R(T)) + \dim(R(U))$.

   (c) Replace $T$ and $U$ above with matrices $A$ and $B$ respectively.

8. Let $E_1, ..., E_k$ be $n \times n$ elementary matrices such that $AE_1...E_k = D$ where $D$ is the reduced row echelon form of $A$ (the main diagonal being all 1's, and the rest of the entries 0). Now, multiply on the right by $D^t$ (the transpose of $D$). Since $DD^t = I_m$, we are done. $B = E_1...E_k D^t$.