MAT 240F - Problem Set 7

Due Thursday, November 20th

Questions 1, 3 b), 4 c) and 5 will be marked.

1. Let V_1, V_2, W_1 and W_2 be vector spaces over a field F. Let $T \in \mathcal{L}(V_1, V_2)$, $U_1 \in \mathcal{L}(W_1, V_1)$ and $U_2 \in \mathcal{L}(V_2, W_2)$. Suppose that nullity$(T)$ is finite and U_1 and U_2 are isomorphisms. (For full marks, do not assume that the vector spaces V_j and W_j are finite-dimensional.) Prove that nullity$(U_2TU_1) =$ nullity(T).

2. Let F be a field and let n be an integer such that $n \geq 2$. Suppose that A, $B \in M_{n\times n}(F)$. We say that A is similar to B if there exists an invertible matrix $C \in M_{n\times n}(F)$ such that $A = C^{-1}BC$.
   a) Show that if A is similar to B, then rank$(A) =$ rank(B).
   b) Show that if A is similar to B, then A is invertible if and only if B is invertible.
   c) Suppose that A and B are invertible. Show that A is similar to B if and only if A^{-1} is similar to B^{-1}.
   d) Show that if A is similar to B, then A^m is similar to B^m for all positive integers m.
   d) Show that if A is similar to B and $A^3 = -A$, then $B^3 = -B$.

3. For each $T \in \mathcal{L}(V, W)$ as defined below, find $T^{-1}(y)$ for each vector $y \in W$.
   a) Let $V = W = P_2(\mathbb{C})$ and let $(Tf)(x) = f(ix) - f(x-1) + f(0)$, $f \in P_2(\mathbb{C})$.
   b) Let $V = P_3(\mathbb{R})$ and $W = M_{2\times 2}(\mathbb{R})$, and let
   $$T(ax^3 + bx^2 + cx + d) = \begin{pmatrix} d+c & -a \\ -b & -c \end{pmatrix}, \qquad a,\ b,\ c,\ d \in \mathbb{R}.$$
   c) Let $V = W = \mathbb{F}_5^3$ and
   $$T(\,a,\ b,\ c\,) = (\,(2a+b+3c)\bmod 5,\ (a+b)\bmod 5,\ (a+b+2c)\bmod 5\,), \qquad a,\ b,\ c \in \mathbb{F}_5.$$

4. Compute rank(T) for each linear transformation T. Explain your answer fully.
   a) Let $T : M_{2\times 2}(\mathbb{C}) \to \mathbb{C}^4$ be defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\,a + i\,c + d, 2i\,a - b + i\,d, a - 3\,c, b - (1+3i)c + (1-i)d\,)$, a, b, c and $d \in \mathbb{C}$.
   b) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(a + bx + cx^2) = a + 2b + c + (a + 3b + 4c)x + (2a + 3b - c)x^2$, $a,\ b,\ c \in \mathbb{R}$.
   c) Let $T : \mathbb{R}^4 \to \mathbb{R}^5$ be defined by $T(a, b, c, d) = (4a - b + c, b + 3d, 3a + c + d, c - d, b + c + 2d)$, $a,\ b,\ c,\ d \in \mathbb{R}$.

5. Suppose that A, $B \in M_{n\times n}(F)$ are invertible. Prove that it is possible to transform A into B using elementary row operations. (*Note*: Elementary column operations should not be used.)

6. Suppose that A, $B \in M_{m\times n}(F)$, and rank$(A) =$ rank(B). Prove that there exist invertible matrices $P \in M_{m\times m}(F)$ and $Q \in M_{n\times n}(F)$ such that $B = PAQ$.

7. #14, §3.2.

8. # 21, §3.2.