MAT 240, Fall 2008
Solutions to Problem Set 6

1.

$$A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix} \quad A^tB = \begin{pmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{pmatrix} \quad BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$$
$$CB = \begin{pmatrix} 27 & 7 & 9 \end{pmatrix} \quad CA = \begin{pmatrix} 20 & 26 \end{pmatrix}$$

2. (a) Assume that UT is one-to-one. If for some $v_1, v_2 \in V$, $T(v_1) = T(v_2)$, then

$$\begin{aligned} U(T(v_1)) &= U(T(v_2)) \\ \Rightarrow UT(v_1) &= UT(v_2) \\ \Rightarrow v_1 &= v_2 \text{ (since UT is one-to-one).} \end{aligned}$$

Therefore, T is one-to-one. However, U does not need to be one-to-one. Indeed, consider the following example. Let $V = \mathbb{R}^2$, $W = \mathbb{R}^3$ and $Z = \mathbb{R}^2$, and let T and U be given by the image of standard basis vectors,

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$U\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; U\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; U\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then, UT is the identity map, so it is one-to-one. However, U is not one-to-one.

(b) Assume that UT is onto, and let $z \in Z$. Then, there exists $v \in V$ such that $UT(v) = z$, i.e. $U(T(v)) = z$. Let $w = T(v) \in W$. Then, we have that $U(w) = z$, so U is onto. However, T does not need to be onto. Look at the example in part a).

3. (a) $\Rightarrow$: Assume that $T^2 = -T$, and let $x \in R(T)$. Then, there exists $v \in V$ such that $x = T(v)$. So $T(x) = T(T(v)) = T^2(v) = -T(v) = -x$.
$\Leftarrow$: Assume that for all $x \in R(T)$, $T(x) = -x$. Let $v \in V$. Then, $x = T(v) \in R(T)$, so $T^2(v) = T(x) = -x = -T(v)$.

(b) Assume that $T^2 = -T$. Let $x \in N(T) \cap R(T)$. Since $x \in R(T)$, there exists $v \in V$ such that $T(v) = x$. Also, $x \in N(T)$, implying that $T(x) = 0$, ie. $0 = T(x) = T(T(v)) = T^2(v) = -T(v) = -x$. Hence, $x = 0$.

(c) $\Rightarrow$: Assume that $dim(V) = n$ and $T^2 = -T$. Let $\{x_1, ..., x_m\}$ be a basis of $N(T)$. Let $\{y_1, ..., y_r\}$ be a basis of $R(T)$ (note, $r = rank(T)$). By Dimension Theorem, $n = r + m$. Since $R(T) \cap N(T) = \{0\}$ by part (b), we have that $\{y_1, ..., y_r, x_1, ..., x_m\}$ is a linearly independent set (proved in previous homeworks) of order n, so it is a basis of V. Moreover, by part (a), $T(y_i) = -y_i$ for all $i = 1, \ldots, r$. Therefore,

$$[T]_\beta = \begin{pmatrix} -I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Second method which does not use part a) and b):
We have that $N(T)$ is finite dimensional vector space with some basis $\{x_1, x_2, \ldots, x_m\}$. By Dimension Theorem, we know that $rank(T) = dim(V) - nullity(T)$, so $r = n - m$. We can extend set $\{x_1, x_2, \ldots, x_m\}$ to an ordered basis of V, $\{y_1, y_2, \ldots, y_r, x_1, x_2, \ldots, x_m\}$. Now, $R(T) = \text{span}(T(\beta)) = \text{span}(\{T(y_1), \ldots, T(y_r), 0, 0, \ldots, 0\}) = \text{span}(\{T(y_1), \ldots, T(y_r)\})$. Since $rank(T) = r$, vectors $T(y_1), \ldots, T(y_r)$ are linearly independent. We claim that $\beta = \{T(y_1), T(y_2), \ldots, T(y_r), x_1, x_2, \ldots, x_m\}$ is a linearly independent set in V. Indeed, let $a_i \in F$, $b_j \in F$, $i = 1, \ldots, r$, $j = 1, \ldots, m$ be such that

$$\begin{aligned}
a_1T(y_1) + \cdots + a_rT(y_r) + b_1x_1 + \cdots + b_mx_m &= 0 \Rightarrow \\
T(T(a_1y_1 + \cdots + a_ry_r) + b_1x_1 + \cdots + b_mx_m) = T(0) &= 0 \Rightarrow \\
T^2(a_1y_1 + \cdots + a_ry_r) + b_1T(x_1) + \cdots + b_mT(x_m) &= 0 \Rightarrow \\
-T(a_1y_1 + \cdots + a_ry_r) &= 0 \Rightarrow \\
-a_1T(y_1) - \cdots - a_rT(y_r) &= 0 \Rightarrow \\
a_1 = \cdots = a_r &= 0 \Rightarrow \\
b_1x_1 + \cdots + b_mx_m &= 0 \Rightarrow \\
b_1 = \cdots = b_m &= 0.
\end{aligned}$$

Also, $|\beta| = r + m = n$, so β is a basis of V. Moreover, since $T^2 = -T$, so $T(T(y_i)) = T^2(y_i) = -T(y_1)$, we have that

$$[T]_\beta = \begin{pmatrix} -I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Note, $r = rank(T)$.
$\Leftarrow$: Assume that there exists a finite basis β of V such that

$$[T]_\beta = \begin{pmatrix} -I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then,

$$[T^2]_\beta = [T]_\beta^2 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = -[T]_\beta.$$

Hence, $T^2 = -T$. Moreover, $rank(T) = rank([T]) = r$.

4. (a) Transformation T is invertible if and only if it is one-to-one and onto (that is bijective). To check if T is one-to-one, let

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 5a & 4b + c \\ 4c + b & 5d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We get that $a = b = c = d = 0$, so T is one-to-one. This also implied that T is onto, since by dimension theorem $rank(T) = 4 - 0 = dim(M_{2\times 2}(\mathbb{R})$. Since T is a bijection, it is invertible.

Now, we will try to find a preimage of a matrix $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, ie.

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 5a & 4b+c \\ 4c+b & 5d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

We get that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a'}{5} & \frac{4b'-c'}{15} \\ \frac{4c'-b'}{15} & \frac{d'}{5} \end{pmatrix}.$$

Let

$$T^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a}{5} & \frac{4b-c}{15} \\ \frac{4c-b}{15} & \frac{d}{5} \end{pmatrix}.$$

We easily check that

$$T\left(T^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = T\begin{pmatrix} \frac{a}{5} & \frac{4b-c}{15} \\ \frac{4c-b}{15} & \frac{d}{5} \end{pmatrix} = \begin{pmatrix} 5\frac{a}{5} & \frac{16b-4c+4c-b}{15} \\ \frac{16c-4b+4b-c}{15} & 5\frac{d}{5} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Similarly, $T^{-1}(T(A)) = A$.

(b) Since $T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$, T is not one-to-one, and hence it is not invertible.

(c) Let $g(x) \in P(\mathbb{C})$. If there exists $f \in P(\mathbb{C})$ such that $T(f) = g$, then $f(2x+i) = g(x)$. Let $t = 2x + i$, so $x = \frac{t-i}{2}$. Then, $f(t) = g(\frac{t-i}{2})$. Let $T^{-1}(f(x)) = f(\frac{t-i}{2})$. We can easily check that $T \circ T^{-1}(f(x)) = T^{-1} \circ T(f(x)) = f(x)$.

(d) Let $U' \in \mathcal{L}(V_1, V_2)$. If there exists $U \in \mathcal{L}(V_1, V_2)$, such that $T(U) = U'$, then $T_2UT_1 = U'$, so $U = T_2^{-1}U'T_1^{-1}$. Let $T^{-1}(U) = T_2^{-1}U'T_1^{-1}$. We can check that $T \circ T^{-1}(U) = T^{-1} \circ T(U) = U$.

5. For the parts of this question in which the subspaces are finite dimensional, it is enough to check if there dimensions are equal. If they are, subspaces are isomorphic. Otherwise, they are not isomorphic.

(a) To determine $dim(V)$, let $A \in M_{3\times 3}(\mathbb{C})$, $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$. Then, if $A \in V$, we have that

$$\begin{aligned}
A &= \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & j \end{pmatrix} = A^t \Rightarrow \\
d &= b,\ g = c,\ h = f \Rightarrow \\
A &= \begin{pmatrix} a & b & c \\ b & e & f \\ c & f & j \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \\
&+ e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + j\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Hence, $dim(V) = 6$.

Similarly, we can determine that if $A \in W$, $A = \begin{pmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{pmatrix}$, so $dim(W) = 3$. Since $dim(V) = 6 \neq 3 = dim(W)$, V and W are not isomorphic.

(b) We know that $dim(W) = 4$. If $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in V$, then

$$\begin{aligned} f(x) &= a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = \\ &= -a_5x^5 - a_4x^4 - a_3x^3 - a_2x^2 - a_1x - a_0 = -f(x) \Rightarrow \\ & \quad a_5 = a_3 = a_1 = 0. \end{aligned}$$

So, $f(x) = a_4x^4 + a_2x^2 + a_0$, and $dim(V) = 3$. Therefore, V is not isomorphic to W.

(c) Since these are infinite dimensional spaces, we need to construct an isomorphism between them. If $f(x) \in W$, then $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x - (a_n + a_{n-1} + \cdots + a_1) = a_n(x^n - 1) + a_{n-1}(x^{n-1} - 1) + \cdots + a_1(x-1)$. Therefore, $W = \text{span}(\{x^n - 1 \mid n \in \mathbb{N}, n \neq 0\})$. Let $T : V \to W$ be $T(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) = a_n(x^{n+1} - 1) + a_{n-1}(x^n - 1) + \cdots + a_0(x-1)$ and $T^{-1} : W \to V$ be $T^{-1}(a_n(x^n - 1) + a_{n-1}(x^{n-1} - 1) + \cdots + a_1(x-1)) = a_nx^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$. Check that $T \circ T^{-1}(f) = f$, for $f \in W$, and $T^{-1} \circ T(g) = g$, for $g \in V$. Hence, V and W are isomorphic.

(d) We have that $dim(V) = dim(P_2(\mathbb{C})) \cdot dim(M_{2\times 2}(\mathbb{C}) = 3 \cdot 4 = 12$. Similarly, $dim(W) = 6 \cdot 2 = 12$. Since $dim(V) = dim(W)$, V is isomorphic to W.

(e) Let $\Phi_\beta : \mathcal{L}(V) \to M_{3\times 3}(F)$ where $\Phi_\beta(T) = [T]_\beta$. We know that Φ_β is invertible. Therefore, $dim(V) = dim(\{[T]_\beta \in M_{3\times 3}(F) \mid [T]_\beta \text{ is diagonal }\})$. Then, $[T]_\beta = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ for some $a, b, c \in F$. That is,

$$[T]_\beta = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, $dim(V) = 3$.

On the other hand, if $T \in W$, then for some $a, b, c \in F$,

$$[T]_\beta = \begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix} = a \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Therefore, $dim(W) = 3$. Since $dim(V) = dim(W)$, V is isomorphic with W.

6. (a) We shall prove this statement by contradiction. Assume that both $T + I_V$ and $T - I_V$ are invertible. Then, we have that $(T + I_V) \circ (T - I_V) = T^2 - T \circ I_V + I_V \circ T - I_V^2 = T^2 - T + T - I_V = T^2 - I_V = 0$, which is the zero map, and it is not invertible. However, in general, a composition of invertible maps is invertible. Indeed, if f and g are invertible maps, then $(f \circ g) \circ (g^{-1} \circ f^{-1}) = f \circ (g \circ g^{-1}) \circ f^{-1} = f \circ f^{-1} = id$ and $(g^{-1} \circ f^{-1}) \circ (f \circ g) = g^{-1} \circ (f^{-1} \circ f) \circ g = g^{-1} \circ g = id$. Hence, we get a contradiction.

(b) Let $dim(V) = n$, and let β be a basis of V. Then let T be a map given by the following $n \times n$ matrix:

$$[T]_\beta = \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & 0 & & & & 1 & 0 \\ \cdot & & & & 1 & 0 & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & \cdot & & & & \cdot \\ 0 & \cdot & & & & & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix}.$$

That is, $[T]_\beta = (a_{ij})$ where $a_{ij} = 1$ if $i+j = n+1$, otherwise $a_{ij} = 0$, for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Then, $[T^2]_\beta = [T]_\beta^2 = I_n$, so $T^2 = I_V$.