MAT 240 - Problem Set 6

Due Thursday, November 13th

Questions 3, 4a), and 5e) will be marked.

1. #2 b) §2.3.
2. #12 §2.3.
3. Let V be a vector space over a field F and let $\mathcal{L}(V)$ be the vector space of linear transformations from V to V. Suppose that $T \in \mathcal{L}(V)$. (For parts a) and b), do not assume that V is finite-dimensional.)
   a) Prove that $T^2 = -T$ if and only if $T(x) = -x$ for all $x \in R(T)$.
   b) Suppose that $T^2 = -T$. Prove that $N(T) \cap R(T) = \{ \mathbf{0} \}$.
   c) Assume that V is finite-dimensional. Prove that $T^2 = -T$ if and only if there exists an ordered basis of V such that
   $$[T]_\beta = [T]_\beta^\beta = \begin{pmatrix} -I_r & 0 \\ 0 & 0 \end{pmatrix},$$
   where $r = \text{rank}(T)$, I_r is the $r \times r$ identity matrix, and each 0 is a zero matrix of the appropriate size.
4. For each of the following linear transformations T, find T^{-1} if T is invertible. If T is not invertible, explain why not. (If T is invertible, be sure to explain why both TT^{-1} and $T^{-1}T$ are identity transformations.)
   a) $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$, $T(A) = 4A + A^t$, $A \in M_{2\times2}(\mathbb{R})$.
   b) $T : M_{2\times2}(\mathbb{C}) \to P_3(\mathbb{C})$ defined by
   $$T\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (a_{11} + a_{12})x^3 + (i\,a_{12} - a_{11})x^2 + (a_{11} - a_{21})x + a_{21}.$$
   c) $T : P(\mathbb{C}) \to P(\mathbb{C})$ defined by $T(f)(x) = f(2x + i)$, $f(x) \in P(\mathbb{C})$.
   d) Let $T_1 : V_1 \to V_1$ and $T_2 : V_2 \to V_2$ be invertible linear transformations. Assume that V_1 and V_2 are vector spaces over the same field F, but do not assume that they are finite-dimensional. Define $T : \mathcal{L}(V_1, V_2) \to \mathcal{L}(V_1, V_2)$ by $T(U) = T_2UT_1$, $U \in \mathcal{L}(V_1, V_2)$.
5. In each part, determine whether the vector spaces V and W are isomorphic. Justify your answers.
   a) Let $V = \{ A \in M_{3\times3}(\mathbb{C}) \mid A = A^t \}$ and $W = \{ A \in M_{3\times3}(\mathbb{C}) \mid A = -A^t \}$.
   b) Let $V = \{ f(x) \in P_5(\mathbb{C}) \mid f(x) = f(-x) \}$ and $W = P_3(\mathbb{C})$.
   c) Let $V = P(\mathbb{R})$ and $W = \{ f(x) \in P(\mathbb{R}) \mid f(1) = 0 \}$.
   d) Let $V = \mathcal{L}(P_2(\mathbb{C}), M_{2\times2}(\mathbb{C}))$ and $W = \mathcal{L}(M_{2\times3}(\mathbb{C}), \mathbb{C}^2)$.
   e) Let V_0 be an 3-dimensional vector space over a field F. Let $\beta = \{ x_1, x_2, x_3 \}$ be an ordered basis for V_0. Define
   $$V = \{ T \in \mathcal{L}(V_0) \mid [T]_\beta \text{ is a diagonal matrix.} \}$$
   $$W = \{ T \in \mathcal{L}(V_0) \mid T(x_1) = T(x_2) = T(x_3) \}.$$
   Note: A matrix $A = (a_{ij}) \in M_{3\times3}(F)$ is a diagonal matrix if $a_{ij} = 0$ whenever $i \neq j$.
6. Let I_V be the identity transformation on V: $I_V(x) = x$ for all $x \in V$. Let $T : V \to V$ be a linear transformation such that $T^2 = I_V$.
   a) Prove that at least one of $T + I_V$ and $T - I_V$ is not invertible. (Do not assume that V is finite-dimensional).
   b) For this part, assume that V is finite-dimensional and $\dim V \geq 2$. Prove that there exists at least one linear transformation $T : V \to V$ such that $T^2 = I_V$ and $T \neq \pm I_V$.