MAT240 Problem Set 5 Solutions

November 18, 2008

Note: There is often more than one solution to a given problem, and so do not interpret these solutions as the only possible set. Also, only partial solutions are given for some problems when appropriate. Finally, if you see any errors, you should let your TA know.

1. (a) Let f be the zero polynomial. Then, $T(f)(x) = xf(x^2+1) + f(1)(x^3-1) = 0+0 = 0$. If $f, g \in P(F)$, then, for any $x \in F$,

$$\begin{aligned} T(f+g)(x) &= x(f+g)(x^2+1) + (f+g)(1)(x^3-1) \\ &= xf(x^2+1) + f(1)(x^3-1) + xg(x^2+1) + g(1)(x^3-1) \\ &= T(f)(x) + T(g)(x). \end{aligned}$$

Finally, if $c \in F$, then,

$$\begin{aligned} T(cf)(x) &= x(cf)(x^2+1) + (cf)(1)(x^3-1) \\ &= c(xf(x^2+1) + f(1)(x^3-1)). \end{aligned}$$

Thus, T is a linear transformation.

(b) This is a linear transformation. The only part that is not clear is checking whether $T(A+B) = T(A) + T(B)$, but the fact that $(A+B)^t = A^t + B^t$ is all that is needed.

(c) Let $f(x) = x^2$. Then, $T(2f)(x) = (2f(x))^2 = 4(f(x))^2 \neq 2(f(x))^2 = 2T(f)(x)$, and so T is not a linear transformation.

(d) This is a linear transformation.

(e) This is a linear transformation. The proof is similar to that of (d).

(f) Consider the vector $2x_1 \in F^3$. Then, $T(2x_1) = (4,0) \neq (2,0) = 2(1,0) = 2T(x_1)$. Hence, T is not a linear transformation.

2. The set $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}(\mathbb{R})$. In particular, any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$ can be expressed as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -c\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + (a-b-c)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (b+c)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (-a+b+c+d)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

So,

$$\begin{aligned}
T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= T\left(-c\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + (a-b-c)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (b+c)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (-a+b+c+d)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= -cT\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) + (a-b-c)T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&+ (b+c)T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) + (-a+b+c+d)T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= -c(x^2-x) + (a-b-c)(3x) + (b+c)(x^2+4) + (-a+b+c+d)(-x^2) \\
&= (a-c-d)x^2 + (3a-3b-2c)x + 4(b+c).
\end{aligned}$$

3. (a) T is linear in each component, and so is a linear transformation.
   (b) $\dim(R(T)) = 3$, and so $R(T) = \mathbb{C}^3$. Thus, there are no conditions on $(a,b,c) \in \mathbb{C}^3$ for this to be in the range. A basis of $R(T)$ is $\{T(1,0,0), T(0,1,0), T(0,0,1)\} = \{(1,2,-1), (-1,1,-2), (2,0,2)\}$.
   (c) For $T(a,b,c) = (0,0,0)$, we must have $(a,b,c) = (0,0,0)$, which is immediate from the dimension theorem and part (b). Thus, the empty set serves as the basis for $N(T)$.

4. (a) A basis for $R(T)$ is
$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}.$$
   A basis for $N(T)$ is
$$\left\{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}.$$
   (b) Since any function of the form $f(x) = cx$ for some constant $c \in \mathbb{R}$ completely describes any element of $N(T)$, a basis for $N(T)$ is $\{x\}$. The range can be shown to be $(x^2-1)(x-1)P(\mathbb{R})$, which is spanned by $\{(x^2-1)(x-1), (x^2-1)(x-1)x, (x^2-1)(x-1)x^2, ...\}$.

5. (a) $0 \in V_1$ and T is a linear transformation, and so $T(0) = 0 \in W_1$. If $u, v \in W_1$, then there exist $u', v' \in V_1$ such that $T(u') = u$ and $T(v') = v$. Now, $u' + v' \in V_1$, and so $T(u'+v') \in W_1$. But, $T(u'+v') = T(u') + T(v') = u+v$. Finally, let $c \in \mathbb{F}$ and $v \in W_1$. Then, again, there is some $v' \in V_1$ such that $T(v') = v$. Then, since $cv' \in V_1$ we have $T(cv') \in W_1$. But, $T(cv') = cT(v') = cv$.
   (b) Let T_1 be the restriction of T to V_1, that is, $T_1 : V_1 \to W_1$ such that $T_1(x) = T(x)$ for all $x \in V_1$. Then T_1 is a linear transformation on V_1, and so we have from the dimension theorem the following:
$$\dim(V_1) = \dim(N(T_1)) + \dim(W_1),$$
   recalling that W_1 is defined to be $R(T_1)$. So, the problem is reduced to $N(T_1) = \{0\}$ if and only if $V_1 \cap N(T) = \{0\}$. Thus, in this light, assume $N(T_1) = 0$. This means that if $x \in V_1$ such that $T_1(x) = 0$, then $x = 0$. Since $T_1(x) = T(x)$, this is exactly saying that the only element of the null space of T in V_1 is 0, or $V_1 \cap N(T) = \{0\}$. Conversely, if $V_1 \cap N(T) = \{0\}$, then the only $x \in V_1$ such that $0 = T(x) = T_1(x)$ is $x = 0$. Thus, $N(T_1) = \{0\}$.

6. (a) Assume that T takes linearly independent sets to linearly independent sets. Then, if $\{v_1, ..., v_n\}$ is any finite linearly independent set, $\{T(v_1), ..., T(v_n)\}$ is also a finite linearly independent set. So, let $x, y \in V$ such that $T(x) = T(y)$. Let $\{v_1, ..., v_n\}$ be a finite linearly independent subset of V such that $x = a_1v_1 + ... + a_nv_n$ and $y = b_1v_1 + ... + b_nv_n$ (for some $a_i, b_i \in \mathbb{F}$). Then,
$$T(a_1v_1 + ... + a_nv_n) = T(b_1v_1 + ... + b_nv_n).$$

Then,

$$\begin{aligned}0 &= T(a_1v_1 + ... + a_nv_n) - T(b_1v_1 + ... + b_nv_n)\\ &= T((a_1 - b_1)v_1 + ... + (a_n - b_n)v_n)\\ &= (a_1 - b_1)T(v_1) + ... + (a_n - b_n)T(v_n).\end{aligned}$$

Since $\{T(v_1), ..., T(v_n)\}$ is linearly independent, $a_i - b_i = 0$ for each i. Thus, $x = y$. Conversely, if T is one-to-one, and $T(x) = 0$, then $x = 0$. So, if $x = a_1v_1 + ... + a_nv_n$ where $\{v_1, ..., v_n\}$ is a linearly independent set, then $0 = T(x) = a_1T(v_1)+...+a_nT(v_n)$ implies that $a_1v_1+...+a_nv_n = 0$, in which case $0 = a_1 = ... = a_n$, and so $\{T(v_1), ..., T(v_n)\}$ is linearly independent.

(b) If S is linearly independent, then so is $T(S)$ by part (a). Conversely, if $\{s_1, ..., s_n\}$ is any finite subset of S then

$$0 = a_1T(s_1) + ... + a_nT(s_n)$$

implies that $a_1 = ... = a_n$. By the linearity of T, this implies that $\{s_1, .., s_n\}$ is linearly independent, and so S is linearly independent.

(c) The fact that T is onto means that $R(T) = W$. So, using the dimension theorem, it suffices to show that $N(T) = \{0\}$. Assume not. Then, there is some $x \in V$ such that $x \neq 0$ and $T(x) = T(0) = 0$. But this contradicts T being one-to-one.

7. Assume $R(T) \cap N(T) = \{0\}$. Let $x \in V$ such that $T(T(x)) = 0$. Then $T(x) \in N(T)$. But clearly $T(x) \in R(T)$ and so $T(x) \in R(T) \cap N(T)$. Thus, by our assumption, $T(x) = 0$. Conversely, assume $T(T(x)) = 0$ implies that $T(x) = 0$, and let $y \in R(T) \cap N(T)$. Then, since $y \in R(T)$, there is some $x \in V$ such that $y = T(x)$. Since $y \in N(T)$, $T(y) = 0$, which is the same as saying $T(T(x)) = 0$. By our assumption, $0 = T(x) = y$.

8. (a) Take $T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2 + x_3, x_1 - x_2 + x_4, -2x_1 - x_3 - x_4, 2x_2 - x_3 + x_4, x_5)$. Then, $R(T)$ is what we wanted, and as for the null space, setting $(x_1 + x_2 + x_3, x_1 - x_2 + x_4, -2x_1 - x_3 - x_4, 2x_2 - x_3 + x_4, x_5) = (0, 0, 0, 0, 0)$ we can see that any element of the null space must satisfy the properties required of elements in $R(T)$.

(b) Assume that there is a linear transformation $T : V \to V$ such that $N(T) \subseteq R(T)$. Then,

$$\begin{aligned}\dim(V) &= \dim(N(T)) + \dim(R(T)) && \text{(dimension theorem)}\\ &\leq \dim(R(T)) + \dim(R(T)).\end{aligned}$$

Conversely, assume W is a subspace of V such that $\dim(W) \geq \dim(V)/2$. Let $\{v_1, ..., v_m\}$ be a basis for W (where $m = \dim(W)$), and let $\{v_1, ..., v_n\}$ be an extension of this basis to a basis of V (where $\dim(V) = n$). Then, consider the transformation defined by $T(v_k) = v_{k-n+m}$ for $k = n - m + 1, ..., n$, and $T(v_l) = 0$ for $l = 1, ..., n - m$. Extend this linearly over all finite linear combinations, and we obtain a linear map $T : V \to V$. Then, $N(T) = \text{span}(\{v_1, ..., v_{n-m}\})$ and $R(T) = \text{span}(\{v_1, ..., v_m\})$. Thus, we have $N(T) \subseteq R(T)$ and $R(T) = W$.

9. (a) This is both one-to-one and onto. (This in fact is the identity transformation on $P(\mathbb{R})$.)

(b) This is one-to-one but not onto: all elements of the range must be divisible by $(x + 1)$.

10. (a) Let $T(x) = a_1y_1 + ... + a_ry_r$. Choose $c_i = -a_i$ for $i = 1, ..., r$. Then,

$$\begin{aligned}T(x + c_1z_1 + ... + c_rz_r) &= T(x) + c_1y_1 + ... + c_ry_r\\ &= a_1y_1 + ... + a_ry_r - a_1y_1 - ... - a_ry_r\\ &= 0.\end{aligned}$$

(b) Consider x as described in part (a). What we showed is that x can be expressed as a sum of a vector in $N(T)$ and a vector in $\text{span}(\{z_1, ..., z_r\})$. Since $N(T) = \text{span}(\{x_1, ..., x_d\})$, we have that $x \in \text{span}(\{x_1, ..., x_d, z_1, ..., z_r\})$. Since x is any vector in V, we have that V is spanned by $d + r$ vectors, and so $\dim(V) \leq d + r$.

11. (a) We can identify the vector space $P_2(\mathbb{C})$ with $\mathbb{R}^3$ and $M_{2\times 2}(\mathbb{C})$ with $\mathbb{R}^4$. We get the following:

$$T(1) = \begin{bmatrix} 3 & i \\ 1 & 0 \end{bmatrix}, \quad T(x-1) = \begin{bmatrix} 3(-i-1) & -i \\ i-1 & 1 \end{bmatrix}, \quad T(x^2 - ix) = \begin{bmatrix} -6 & 0 \\ 0 & -i \end{bmatrix}.$$

Converting these results to the basis γ, we have:

$$[T(1)]_\gamma = \begin{bmatrix} (i-1)/2 & (i+1)/2 \\ 3 & 0 \end{bmatrix}, \quad [T(x-1)]_\gamma = \begin{bmatrix} 1/2 - i & -1/2 \\ -3(i+1) & 1 \end{bmatrix}, \quad [T(x^2-ix)]_\gamma = \begin{bmatrix} 0 & 0 \\ -6 & -i \end{bmatrix}.$$

Thus, we can conclude that

$$[T]_\beta^\gamma = \begin{bmatrix} (i-1)/2 & 1/2 - i & 0 \\ (i+1)/2 & -1/2 & 0 \\ 3 & -3(i+1) & -6 \\ 0 & 1 & -i \end{bmatrix}.$$

(b) $T(a, b, c) = (a - b, 3a + b - c, -2a + c)$.

12. $T(x^2 + x + i) = (i/2, i, -i/2, 2 - i)$.

13.

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

14. Let $a, b \in \mathbb{F}$ such that $aT + bU = 0$. That is, for any $v \in V$, $aT(v) + bU(v) = 0$. Then, since T and U are linear, $T(av) = U(-bv)$. But then, we have $aT(v), -bU(v) \in R(T) \cap R(U)$. But since $R(T) \cap R(U) = \{0\}$, then $aT(v) = -bU(v) = 0$. Since v is arbitrary and T, U are both nonzero linear transformations, we must have that $a = b = 0$, and so T, U are linearly independent in $\mathcal{L}(V, W)$.

15. (a) Let $\gamma' = \{w_1, ..., w_r\}$ be a basis for $R(T) \subseteq W$, and let $\gamma = \{w_1, ..., w_n\}$ be a basis of W that extends γ'. Choose $v_1, ..., v_r \in V$ such that $T(v_i) = w_i$ for each $i = 1, ..., r$. Then it is easy to show that $\{v_1, ..., v_r\}$ is linearly independent, and extending this set to a basis $\{v_1, ..., v_n\}$ of V, which we shall denote β. Then, $T(v_i) = 0$ for each $i = r + 1, ..., n$, and the matrix representation $[T]_\beta^\gamma$ of T is the diagonal matrix desired.

(b) The number of nonzero entries in A is r, which is the dimension of $R(T)$, which in turn is the rank of T.