MAT 240 - Problem Set 5

Due Thursday November 6th

Questions 1a), 8, and 11a) will be marked.

1. Determine whether the function $T : V \to W$ is a linear transformation.
   a) Let $V = W = P(F)$, where F is a field. Define $T(f)(x) = xf(x^2 + 1) + f(1)(x^3 - 1)$, $f \in V$.
   b) Let $V = W = M_{2\times 2}(\mathbb{C})$. Define $T(A) = i\,A^t - 4\,A$, $A \in V$. (Here, A^t is the transpose of the matrix A.)
   c) Let $V = P_2(\mathbb{C})$ and $W = P_4(\mathbb{C})$. Define $T(f)(x) = (f(x))^2$, $f \in V$.
   d) Let $V = P_6(F)$ (F a field) and $W = F^3$. Define $T(f) = (a_6, -a_3, a_1 + a_4)$, for $f(x) = a_6x^6 + \cdots + a_1x + a_0 \in V$.
   e) Let V be a finite-dimensional vector space over a field F. Let $n = \dim V$ and let $W = F^n$. Let $\{ x_1, \ldots, x_n \}$ be a basis of V. Define $T(x) = (-c_1, c_2 - c_1, c_3 - c_2, c_4 - c_3, \ldots, c_j - c_{j-1}, \ldots, c_n - c_{n-1})$ for $x = c_1x_1 + c_2x_2 + \cdots + c_nx_n \in V$ ($c_1, \ldots, c_n \in F$).
   f) Let $V = F^3$ and $W = F^2$, where F is a field. Let $\{ x_1, x_2, x_3 \}$ be a basis of V. Define $T(c_1x_1 + c_2x_2 + c_3x_3) = (c_1^2 + c_2, c_2 - c_3)$, c_1, c_2, $c_3 \in F$.

2. Assume that $T : V \to W$ is a linear transformation, where $V = M_{2\times 2}(\mathbb{R})$ and $W = P_2(\mathbb{R})$. Suppose that

$$T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = x^2 - x, \quad T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 3x, \quad T\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = x^2 + 4, \quad T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -x^2.$$

   Determine $T\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for all real numbers a, b, c and d.

3. Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be defined by:

$$T(a, b, c) = (a - b + 2c, 2a + b, -a - 2b + 2c), \qquad a,\ b,\ c \in \mathbb{C}.$$

   a) Verify that T is a linear transformation.
   b) If $(a, b, c) \in \mathbb{C}^3$, what are the conditions on a, b, and c that $(a, b, c) \in R(T)$? (Here, $R(T)$ denotes the range of T). Find a basis for $R(T)$.
   c) If $(a, b, c) \in \mathbb{C}^3$, what are the conditions on a, b, and c that $(a, b, c) \in N(T)$? (Here, $N(T)$ denotes the null space of T). Find a basis for $N(T)$.

4. Find a basis of $N(T)$ and a basis of $R(T)$ for the given linear transformation T.
   a) Let $V = M_{2\times 3}(\mathbb{C})$ and $W = M_{2\times 2}(\mathbb{C})$. Define

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} a_{13} & -2\,a_{11} \\ (1 - i)(a_{12} - a_{21}) & 0 \end{pmatrix}, \quad a_{ij} \in \mathbb{C}, 1 \le i \le 2,\ 1 \le j \le 3.$$

   b) Let $V = W = P(\mathbb{R})$. Define $T(f)(x) = (x^2 - 1)(f(x) - f(1)\,x)$.

5. Let $T : V \to W$ be a linear transformation. Let V_1 be a subspace of V. Define $W_1 = \{ T(x) \mid x \in V_1 \}$.
   a) Prove that W_1 is a subspace of W.
   b) Assume that V is finite-dimensional. Prove that $\dim W_1 = \dim V_1$ if and only if $V_1 \cap N(T) = \{\mathbf{0}\}$.
6. #14, §2.1.
7. Let $T : V \to V$ be a linear transformation. Prove that the following are equivalent:
   (i) $R(T) \cap N(T) = \{\mathbf{0}\}$.
   (ii) For $x \in V$, $T(T(x)) = \mathbf{0}$ implies that $T(x) = \mathbf{0}$.
8. a) Find an example of a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that $N(T) \subset R(T)$ and

$$R(T) = \{ x = (a_1, \ldots, a_5) \in \mathbb{R}^5 \mid a_1 + a_2 + a_3 = 0, \text{ and } a_1 - a_2 + a_4 = 0 \}.$$

   b) Suppose that V is a finite-dimensional vector space. Suppose that W is a subspace of V. Prove that there exists a linear transformation $T : V \to V$ such that $N(T) \subset R(T)$ and $R(T) = W$ if and only if $\dim(W) \geq \dim(V)/2$.
9. Determine whether or not the linear transformation $T : V \to V$ is one-to-one. Also determine whether or not T is onto.
   a) Let $V = P(\mathbb{R})$. Define $T(f)(x) = x\, f'(x) + f(0)$, $f \in V$.
   b) Let V be the vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Define $T(f)(x) = (x + 1)f(x)$, $f \in V$.
10. Let V and W be vector spaces over a field F. Suppose that $T : V \to W$ is a linear transformation that has the property that $N(T)$ and $R(T)$ are finite-dimensional.
    a) Let $\{ x_1, \ldots, x_d \} \subset V$ be a basis of $N(T)$ and let $\{ y_1, \ldots, y_r \} \subset W$ be a basis of $R(T)$. Choose vectors $z_1, \ldots, z_r \in V$ such that $y_j = T(z_j)$, $1 \leq j \leq r$. Let $x \in V$. Prove that there exist $c_1, \ldots, c_r \in F$ such that $x + c_1 z_1 + c_2 z_2 + \cdots + c_r z_r \in N(T)$.
    b) Prove that V is finite-dimensional. (*Hint*: One way to do this is to prove that $V = \operatorname{span}(S)$, where $S = \{ x_1, \ldots, x_d, z_1, \ldots, z_r \}$.)
11. a) Let $T : P_2(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$ be the linear transformation defined by

$$T(f(x)) = \begin{pmatrix} 3\, f(-i) & i\, f(0) \\ f(i) & f'(0) \end{pmatrix}, \qquad f(x) \in P_2(\mathbb{C}).$$

Let $\beta = \{ 1, x - 1, x^2 - i\, x \}$ and

$$\gamma = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Compute the matrix $[T]_\beta^\gamma$ of T with respect to the ordered bases β and γ.

b) Let $\beta = \{\,(1,-1,0),(1,0,-1),(0,0,1)\,\}$ and $\gamma = \{\,(1,0,0),(0,1,0),(0,1,1)\,\}$.

$$A = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 3 & 1 \end{pmatrix} \in M_{3\times 3}(\mathbb{Q}).$$

Let $T : \mathbb{Q}^3 \to \mathbb{Q}^3$ be the linear transformation such that $[T]_\beta^\gamma = A$. Compute $T(a,b,c)$ for all a, b, $c \in \mathbb{Q}$.

12. Let $\beta = \{ix, -1, x + x^2\} \subset P_2(\mathbb{C})$ and

$$\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \subset M_{2\times 2}(\mathbb{C}).$$

Let $T : P_2(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$ be the linear transformation such that

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & -i & 1 \\ 3 & 0 & i \\ 2 & i & -i \\ 1 & 1 & 0 \end{pmatrix}.$$

Compute $T(x^2 + x + i)$. (Show your work.)

13. #10 §2.2.

14. #13 §2.2.

15. Suppose that V and W are finite-dimensional vector spaces over F. Assume that $\dim V = \dim W = n$. Let $T : V \to W$ be a linear transformation.
    a) Prove that there exist ordered bases β for V and γ for W such that the matrix $A = (A_{jk}) = [T]_\beta^\gamma$ satisfes $A_{jk} = 0$ if $j \neq k$ and $A_{jj} = 0$ or 1, for $1 \leq j, k \leq n$.
    b) Let A, β and γ be as in part a). Show that the number of nonzero diagonal entries of A is equal to the rank of T.