MAT240 Problem Set 1 Solutions

October 16, 2008

Note: There is often more than one solution to a given problem, and so do not interpret these solutions as the only possible set. Also, only partial solutions are given for some problems when appropriate. Finally, if you see any errors, you should let your TA know.

1. (a) A basis for W is $\{u, v, w\}$ where $u = (i, -1, 1, 0, 0)$, $v = (-1-i, 1-i, 0, 1, 0)$ and $w = (-2, -4i, 0, 0, 1)$.

   (b) A basis for W is $\{1\}$; that is, W contains only the constants.

   (c) A basis for W is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

   (d) A basis for W is $B = \{i, -2x+1, -2x^2+1, ..., -2x^n+1\}$. Let $a_0, ..., a_n \in \mathbb{C}$ such that

   $$a_0 i + a_1(-2x+1) + ... + a_n(-2x^n+1) = 0.$$

   Then, $-2a_n x^n - 2a_{n-1}x^{n-1} - ... - 2a_1 x + (a_0 i + a_1 + ... + a_n) = 0$. Since we know that $\{1, x, x^2, ..., x^n\}$ is linearly independent, we have that each of $a_n, ..., a_1$ is zero. That leaves $a_0 i = 0$, and so we have $a_0 = 0$ as well. Thus, B is linearly independent. It is a basis since it has $n+1$ elements (which is the dimension of $W = P_n(\mathbb{C})$) and each element f of B clearly satisfies $f(0)f(1) = -1$.

2. (a) S is linearly independent, and thus forms a basis for span(S), and so span(S) has dimension 2.

   (b) The set $B = \{1, x^2\}$ is a basis for span(S), which thus has dimension 2.

   (c) The set $B = \{x+z, y+z\}$ is a basis for span(S), which thus has dimension 2.

   (d) The set $B = \left\{ \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for span(S), which thus has dimension 3. Note that

   $$\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}.$$

   (e) The set $B = \{y, x_1, ..., x_n\}$ is a basis for span(S). Note that since $y \notin \text{span}(\{x_1, ..., x_n\})$, we know that B is linearly independent. The dimension of span(S) is $n+1$.

3. Let u, v, w be distinct vectors of a vector space V defined over a field $\mathbb{F}$. If $\{u, v, w\}$ is a basis for V, then $\{u+v+w, v+w, w\}$ is a basis for V.

   Proof. First we will check linear independence. Let $a, b, c \in \mathbb{F}$ such that $a(u+v+w)+b(v+w)+cw = 0$. Then,

   $$\begin{aligned} 0 &= a(u+v+w) + b(v+w) + cw \\ &= au + (a+b)v + (a+b+c)w. \end{aligned}$$

   However, since $\{u, v, w\}$ is linearly independent, $a = a+b = a+b+c = 0$, which implies that $a = b = c = 0$, and so $\{u+v+w, v+w, w\}$ is linearly independent. It spans a subspace of V of dimension 3. But since $\{u, v, w\}$ is a basis for V, and hence the dimension of V is 3, $\{u+v+w, v+w, w\}$ must be a basis for V. □

4. For a fixed $a \in \mathbb{R}$, the dimension of the subspace $W = \{f \in P_n(\mathbb{R}) \mid f(a) = 0\}$ of $P_n(\mathbb{R})$ is n.

   Proof. For any $f \in W$, by the fundamental theorem of algebra, $f(x) = (x-a)g(x)$ for some $g \in P_{n-1}(\mathbb{R})$. That is, $f(x) = (x-a)(b_0 + b_1x + ... + b_{n-1}x^{n-1})$ where $g(x) = b_0 + b_1x + ... + b_{n-1}x^{n-1}$ and $b_i \in \mathbb{R}$ for $i = 0, ..., n-1$. So,

   $$f(x) = b_0(x-a) + b_1(x-a)x + ... + b_{n-1}(x-a)x^{n-1}.$$

   Note that for any such numbers b_i, we obtain an element of W, and any element of W can be expressed as above. Thus, $W = \text{span}(\{(x-a), (x-a)x, ..., (x-a)x^{n-1}\})$. It is easy to check that $\{(x-a), (x-a)x, ..., (x-a)x^{n-1}\}$ is linearly independent, and so it forms a basis for W, and the dimension of W is n. ☐

5. By definition, if n is odd, then $W_1 \cap P_n(F) = \{a_nx^n + a_{n-2}x^{n-2} + ... + a_1x \mid a_1, ..., a_n \in F\}$ and $W_2 \cap P_n(F) = \{a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + ... + a_0 \mid a_0, ..., a_{n-1} \in F\}$ (and vice versa if n is even). Thus, a basis for $W_1 \cap P_n(F)$ is $\{x, x^3, x^5, ..., x^n\}$, and a basis for $W_2 \cap P_n(F)$ is $\{1, x^2, x^4, ..., x^{n-1}\}$. Hence, the dimensions of the two subspaces are both $\frac{n+1}{2}$. If n is even, then a basis for $W_1 \cap P_n(F)$ is $\{x, x^3, x^5, ..., x^{n-1}\}$, and a basis for $W_2 \cap P_n(F)$ is $\{1, x^2, x^4, ..., x^n\}$. Hence, the dimensions of the two subspaces are $\frac{n}{2}$ and $\frac{n}{2} + 1$, respectively.

6. (a) Assume that there did exist a basis B of $P_n(F)$ such that B did not contain a polynomial of degree n. Then $B \subset P_{n-1}(F)$, and so $\text{span}(B) \subset P_{n-1}(F)$ since $P_{n-1}(F)$ is a subspace of $P_n(F)$. Since B is linearly independent, it has at most $n = \dim(P_{n-1}(F))$ elements. But since B is a basis of $P_n(F)$, it should have $n+1$ elements. This is a contradiction.

   (b) $B = \{1 + x^{n+1}, x + x^{n+1}, ..., x^n + x^{n+1}, x^{n+1}, x^{n+1} + x^{n+2}, x^{n+1} + x^{n+3}, ...\}$ is a basis for $P(F)$.

7. (This is actually the second #6 on the assignment.) Let $\{b_1, ..., b_n\}$ be a basis for V. Then, take $W = \text{span}(\{b_1, ..., b_j\})$. This is a subspace of dimension j.

8. (This is actually #7 on the assignment.)

   (a) Let $\{b_1, ..., b_j\}$ be a basis for W_1. If $b_{j+1} \notin W_1$, then the span of $\{b_1, ..., b_{j+1}\}$ is a $j+1$-dimensional subspace containing W_1. We can continue choosing vectors $b_{j+i} \notin \text{span}(\{b_1, ..., b_{j+i-1}\})$ inductively until we obtain a basis for an n-dimensional subspace, which would be V itself. Now, let $W_2 = \{b_{j+1}, ..., b_n\}$. $\dim(W_2) = n - j$ as desired, and so we just need to verify that $W_1 \cap W_2 = 0$. So, let $x \in W_1 \cap W_2$. Then, $x \in W_1$ and $x \in W_2$. But then, $x = a_1b_1 + ... + a_jb_j = a_{j+1}b_{j+1} + ... + a_nb_n$ for some $a_i \in F$, $i = 1, ..., n$. Thus, $0 = a_1b_1 + ... + a_jb_j - a_{j+1}b_{j+1} - ... - a_nb_n$, and since $\{b_1, ..., b_n\}$ forms a basis for V, $a_1 = ... = a_n = 0$. Thus, $x = 0$, and we are done.

   (b) For any $x \in V$, using the same notation as in the solution to part (a), $x = a_1b_1 + ... + a_nb_n$ for some $a_i \in F$, $i = 1, ..., n$. But, $a_1b_1 + ... + a_jb_j \in W_1$ (call this sum x_1) and $a_{j+1}b_{j+1} + ... + a_nb_n \in W_2$ (call this sum x_2). We have $x = x_1 + x_2$ satisfying the appropriate properties. As for uniqueness for x_1 and x_2, assume that there exist $x_1' \in W_1$ and $x_2' \in W_2$ such that $x = x_1 + x_2 = x_1' + x_2'$. Then,

   $$\begin{aligned} 0 &= x - x \\ &= (x_1 + x_2) - (x_1' + x_2') \\ &= (x_1 - x_1') + (x_2 - x_2'). \end{aligned}$$

   Note that $x_1 - x_1' \in W_1$ and $x_2 - x_2' \in W_2$, and their sum is 0. That means that $x_1 - x_1' = -(x_2 - x_2') = x_2' - x_2$. So, we also have that $x_1 - x_1' \in W_2$ and $x_2 - x_2' \in W_1$. Since $W_1 \cap W_2 = 0$, we conclude that $x_1 - x_1' = x_2 - x_2' = 0$. In other words, $x_1 = x_1'$ and $x_2 = x_2'$.

9. (This is #8 on the assignment.) We will prove this by contradiction. Assume $W_1 \cap W_2 = 0$. If S_1 is a basis for W_1 and S_2 is a basis for W_2, then by #11(a) from Problem Set 3, we know that $S_1 \cup S_2$ is a linearly independent set. But $S_1 \cup S_2$ has $\dim(W_1) + \dim(W_2) > n$ elements, and so it spans a subspace of V (since $S_1 \cup S_2 \subset V$) that has dimension bigger than $n = \dim(V)$. This is absurd, and we have our contradiction. Thus, $W_1 \cap W_2 \neq 0$.