MAT 240 - Problem Set 4

Due Thursday October 16th

Questions 1d), 2c), 2d), and 7 will be marked.

1. In each case, find a basis for the indicated subspace W of the vector space V. In some parts, you must find a basis for W that has certain additional properties. (Be sure to demonstrate that the set you find is linearly independent and spans W.)

   a) Let W be the subspace of $V = \mathbb{C}^5$ consisting of all vectors $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5$ that satisfy
$$-2ix_1 + x_2 - x_3 + (1-i)x_4 = 0$$
$$x_1 + ix_2 - 2x_5 = 0$$

   b) Let $V = P_2(\mathbb{C})$ and $W = \{\, f(x) \in V \mid f(x) = f(x-i) \,\}$.

   c) Let $V = W = M_{2\times 2}(F)$. Find a basis for W that has the property that every element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belonging to the basis satisfies $a + b = 1$.

   d) Let $V = P_n(\mathbb{C})$, where $n \geq 1$. Find a basis for $W = V$ such that every $f(x)$ belonging to the basis satisfies $f(0)f(1) = -1$.

2. In each case below, for the given subset S of V, find a basis for the subspace $\mathrm{span}(S)$ of V, and compute the dimension of $\mathrm{span}(S)$.

   a) Let V be the space of functions from $\mathbb{R}$ to $\mathbb{R}$, and let $S = \{\, f(t) = e^{rt},\ g(t) = e^{st} \,\}$, where r and s are fixed real numbers such that $r \neq s$.

   b) Let $V = P(\mathbb{R})$ and $S = \{\, x^2 - 1,\ x^2 + 1,\ x^2 - 2, x^2 + 2 \,\}$.

   c) Let V be a vector space over the field $\mathbb{R}$ of dimension at least 3 and let $S = \{\, x + z,\ x + y, 3x - y + 4z \,\}$, where x, y and z are distinct vectors in V such that $\{\, x, y, z \,\}$ is linearly independent.

   d) Let $V = M_{2\times 2}(\mathbb{F}_5)$ and let
$$S = \left\{ \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \right\}.$$

   e) Let V be a vector space of dimension at least $n + 1$, where n is a positive integer. Let
$$S = \{\, y+x_1, y+x_2, \ldots, y+x_n \,\} \cup \{\, y+x_1+x_2, y+x_1+x_2+x_3, \ldots, y+x_1+x_2+\cdots+x_n \,\},$$
   where $\{\, x_1, x_2, \ldots, x_n \,\}$ is linearly independent and $y \in V$ does not belong to $\mathrm{span}(\{\, x_1, x_2, \ldots, x_n \,\})$.

3. Question 12 of §1.6.
4. Question 26 of §1.6.
5. Question 27 of §1.6.
6. Let n be a positive integer and let F be a field.

   a) Prove that any basis for $P_n(F)$ must contain a polynomial of degree n.

b) Find a basis for $P(F)$ that has the property that all elements in the basis have degree at least $n+1$. (Be sure to demonstrate that the set you find is linearly independent and spans $P(F)$.)

6. Let $n \geq 2$ and let V be an n-dimensional vector space over a field F. Let j be an integer such that $1 \leq j \leq n-1$. Prove that there exists at least one subspace W of V such that $\dim(W) = j$.

7. Let $n \geq 2$ and let V be an n-dimensional vector space over a field F. Suppose that j is an integer such that $1 \leq j \leq n-1$ and W_1 is a subspace of V such that $\dim(W_1) = j$.

   a) Prove that there exists a subspace W_2 of V such that $W_1 \cap W_2 = \{\mathbf{0}\}$ and $\dim(W_2) = n - j$. (*Hint*: Corollary 2 of Theorem 1.10, or Theorem 1 of the notes posted on the course home page, may be useful.)

   b) Suppose that W_2 is as in part a). Prove that every vector $x \in V$ can be expressed in the form $x = x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$. Show that, given a fixed x, the vectors $x_1 \in W_1$ and $x_2 \in W_2$ are unique.

8. Let V be a vector space over a field F. Suppose that $n \geq 2$ and $\dim(V) = n$. Let W_1 and W_2 be subspaces of V such that $\dim(W_1) + \dim(W_2) > n$. Prove that $W_1 \cap W_2 \neq \{\mathbf{0}\}$. (*Hint*: Let S_1 be a basis for W_1 and let S_2 be a basis for W_2. If you assume that $W_1 \cap W_2 = \{\mathbf{0}\}$, what can you say about properties of the set $S_1 \cup S_2$?)