MAT 240, Fall 2008
Solutions to Problem Set 3

1. Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}$. Then, $x \in \text{span}(\{y, z\})$ if and only if there exist $a, b \in \mathbb{C}$ such that $x = ay + bz$, i.e. if and only if $(x_1, x_2, x_3, x_4) = (b, a + bi, a(1 + i) + b, -ai - bi)$. This is equivalent to $b = x_1$, $a = x_2 - ix_1$, $x_3 = (2 - i)x_1 + (1 + i)x_2$ and $x_4 = (-1 - i)x_1 - ix_2$.

2. We will prove each direction separately.
$\Rightarrow$: Let W be a subspace of V. Then, $\text{span}(W) = \{\sum_{i=1}^{n} a_i w_i \mid a_i \in F, w_i \in W, n \in \mathbb{N}\} \subset W$, since W is a subspace so $\sum_{i=1}^{n} a_i w_i \in W$. Also, $W = \{1 \cdot w \mid w \in W\} \subset \text{span}(W)$. Hense, $\text{span}(W) = W$.
$\Leftarrow$: Assume $\text{span}(W) = W$. We check conditions of subspace test (Theorem 1.3):

   (a) $0 \in W$ since $0 = 0 \cdot w \in \text{span}(W)$ for any $w \in W$ (note we can prove that $0 \cdot w = 0$ in a vector space the same way we proved that in any field F for any $x \in F$, $0 \cdot x = 0$);
   (b) let $w_1, w_2 \in W$, then $w_1 + w_2 \in \text{span}(W) = W$,
   (c) let $c \in F$, $w \in W$, then $cw \in \text{span}(W) = W$.

   Hence, W is a subspace of V.

3. Let V be a vector space over a field F and let S_1 and S_2 be subsets of V. Then it is easy to prove that if $S_1 \subset \text{span}(S_2)$ then $\text{span}(S_1) \subset \text{span}(S_2)$. Indeed, $\text{span}(S_1) = \{\sum_{i=1}^{n} c_i s_i \mid c_i \in \mathbb{F}_5, s_i \in S_1, i \in \mathbb{N}\} = \left\{\sum_{i=1}^{n} c_i \left(\sum_{j=1}^{m} d_j z_j\right) \mid c_i, d_j \in \mathbb{F}_5, z_j \in S_2, i, j \in \mathbb{N}\right\} = \left\{\sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_j z_j \mid c_i, d_j \in \mathbb{F}_5, z_j \in S_2, i, j \in \mathbb{N}\right\} \subset \text{span}(S_2)$. We use this observation in the following problems.

   (a) We have that $S_2 \subset S_1$ which implies that $\text{span}(S_2) \subset \text{span}(S_1)$. On the other hand, if $f \in S_1$, then $f(x) = (x - 1)g(x)$, where $g(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, for some $a_i \in \mathbb{R}$, $i = 0, 1, 2, 3$. So, $f(x) = a_3x^3(x - 1) + a_2x^2(x - 1) + a_1x(x - 1) + a_0(x - 1) \in \text{span}(S_2)$. Hence, $\text{span}(S_1) \subset \text{span}(S_2)$. Therefore, $\text{span}(S_1) = \text{span}(S_2)$.
   (b) Obviously, $z \in S_2$ and $z \notin \text{span}(S_1)$, so $\text{span}(S_2)$ is not a subset of $\text{span}(S_1)$. Note however that $x = (x + z) - z$, so $S_1 \subset \text{span}(S_2)$, and hence $\text{span}(S_1) \subset \text{span}(S_2)$.
   (c) We have that $S_1 \subset S_2$, hence $\text{span}(S_1) \subset \text{span}(S_2)$. We want to prove that $S_2 \subset \text{span}(S_1)$, which implies that $\text{span}(S_2) \subset \text{span}(S_1)$. Then we have that $\text{span}(S_1) = \text{span}(S_2)$.
   Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_2$. If $A \in \text{span}(S_1)$, then there exist $\alpha, \beta, \gamma \in \mathbb{F}_5$ such that

$$A = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} + \beta \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

   Using calculations in $\mathbb{F}_5$ to solve the following system of equations for α and β,

$$\begin{array}{rcrcrcl} \alpha & & & + & 2\gamma & = & a \\ & & 2\beta & + & 3\gamma & = & b \\ & & 3\beta & + & 2\gamma & = & c \\ 4\alpha & & & + & 3\gamma & = & d \end{array}$$

we get that

$$\begin{array}{rcccl} 0 & = & 4a-d & = & 4a+4d \\ 0 & = & 3b-2c & = & 3b+3c. \end{array}$$

Multiplying the first equation by 4 and the second by 2, we get that

$$\begin{array}{rcl} 0 & = & a+d \\ 0 & = & b+c \end{array}$$

Hence, we have one free variable $\gamma \in \mathbb{F}_5$ and $(\alpha, \beta, \gamma) = (a+3\gamma, 3b+\gamma, \gamma)$.

4. (a) Let $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. To show that $\mathrm{span}(S) = V$, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, for some $a, b, c, d \in \mathbb{Q}$. Then,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a-c-d) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, $V \subset \mathrm{span}(S)$. Obviously, $\mathrm{span}(S) \subset V$, hence $\mathrm{span}(S) = V$.

(b) If, $\mathrm{span}(S') = V$, then $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{span}(S')$. That is, there exist $c_i \in \mathbb{Q}$, such that

$$\begin{array}{ccl} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & = & \sum_{i=1}^n c_i \begin{pmatrix} a_{i11} & a_{i12} \\ a_{i21} & a_{i22} \end{pmatrix}, \text{ where } \begin{pmatrix} a_{i11} & a_{i12} \\ a_{i21} & a_{i22} \end{pmatrix} \in S' \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & = & \begin{pmatrix} \sum_{i=1}^n c_i a_{i11} & \sum_{i=1}^n c_i a_{i12} \\ \sum_{i=1}^n c_i a_{i21} & \sum_{i=1}^n c_i a_{i22} \end{pmatrix} \\ 1+0 & = & \sum_{i=1}^n c_i a_{i11} + \sum_{i=1}^n c_i a_{i12} \\ 1 & = & \sum_{i=1}^n c_i (a_{i11} + a_{i12}) \\ 1 & = & \sum_{i=1}^n c_i \cdot 0 \\ 1 & = & 0. \end{array}$$

Contradiction.

5. (a) Take $S = \{x^2 + x, 2x^2 + x, x^2 + x + 1\}$. Then, $ax^2 + bx + c = (-a + 2b - c)(x^2 + x) + (a - b)(2x^2 + x) + c(x^2 + x + 1)$, for any $a, b, c \in \mathbb{F}_5$.

(b) Consider $g(x) = x$, $g \in V$. If $g(x) = \sum_{i=1}^n c_i f_i(x)$, for some $f_i \in S'$ and $c_i \in \mathbb{F}_5$, then $g(1) = \sum_{i=1}^n c_i f_i(1) = \sum_{i=1}^n c_i f_i(3) = g(3)$. However, $g(1) = 1 \neq 3 = g(3)$. Contradiction.

6. (a) Let $v \in \mathrm{span}(S_1 \cap S_2)$. Then $v = \sum_{i=1}^n c_i s_i$ for some $s_i \in S_1 \cap S_2$ and scalars c_i. Since $s_i \in S_1$ for all i, $v \in \mathrm{span}(S_1)$. Also, $s_i \in S_2$ for all i, so $v \in \mathrm{span}(S_2)$. Hence, $v \in \mathrm{span}(S_1) \cap \mathrm{span}(S_2)$.

(b) Consider $V = \mathbb{R}^2$, $S_1 = \{(1,0)\}$, $S_2 = \{2,0)\}$. Then $\mathrm{span}(S_1 \cap S_2) = \mathrm{span}(\emptyset) = \{0\}$ and $\mathrm{span}(S_1) \cap \mathrm{span}(S_2) = \mathrm{span}(S_1)$, i.e. the x-axis.

(c) Let $V = \mathbb{R}^2$, $S_1 = \{(1,0)\}$, $S_2 = \{1,0),(0,1)\}$. Then $\mathrm{span}(S_1 \cap S_2) = \mathrm{span}(\{(1,0)\}) = \mathrm{span}(S_1) \cap \mathrm{span}(S_2)$.

7. (a) Let $a, b, c \in \mathbb{C}$ and the linear combination

$$a\begin{pmatrix} -1+i & 1+i \\ 0 & 1-i \end{pmatrix} + b\begin{pmatrix} i+1 & 1 \\ -1 & i-1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ i & 2 \end{pmatrix} = 0,$$

which gives system of equations

$$\begin{array}{ccccccc} (-1+i)a & + & (1+i)b & & & = & 0 \\ (1+i)a & + & b & + & c & = & 0 \\ & - & b & + & ic & = & 0 \\ (1-i)a & + & (-1+i)b & + & 2c & = & 0. \end{array}$$

Solving this system for b and c, we get $0 \cdot a = 0$, so we can take any $a \in \mathbb{C}$. So, $(a, b, c) = (a, \frac{1-i}{1+i}a, -a)$. In particular, if we take $a = 1+i$, we get a non-trivial solution $(a, b, c) = (1+i, 1-i, -1-i)$. Hence, S is not linearly independent in V.

(b) It is easy to show that if

$$a\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0,$$

then $a = b = c = 0$, so S is a linearly independent set.

(c)

$$\begin{array}{rcl} a(x^5 + x^4 - x^3) + b(x^5 + x^4 - x^2) + c(x^5 + x^4 - x) & = & 0 \\ (a+b+c)x^5 + (a+b+c)x^4 - ax^3 - bx^2 - cx^2 & = & 0. \end{array}$$

So,

$$\begin{array}{rcl} a+b+c & = & 0 \\ a+b+c & = & 0 \\ -a & = & 0 \\ -b & = & 0 \\ -c & = & 0 \end{array}$$

Obviously, $(a, b, c) = (0, 0, 0)$, so S is linearly independent.

8. (a) This statement is the same as: The set $S = \{f, g\}$ is linearly dependent if and only if for all $a, b \in F$, $f(a)g(b) - g(a)f(b) = 0$.
$\Rightarrow$: Assume that S is linearly dependent. Then there exist $(\alpha, \beta) \neq (0, 0)$, such that $\alpha f(x) + \beta g(x) = 0$ for all $x \in F$. Without loss of generality, assume that $\alpha \neq 0$, so there exists α^{-1}. Then, $f(x) = (-\beta)\alpha^{-1}g(x)$. For all $a, b \in F$, $f(a)g(b) = (-\beta)\alpha^{-1}g(a)g(b) = g(a)f(b)$.
$\Leftarrow$: If f and g are identically zero, then $f = g$, so S is linearly dependent. Assume that not both f and g are identically zero. Say, without loss of generality, that there exists $x_0 \in F$ such that $g(x_0) \neq 0$, so there exists $g(x_0)^{-1}$. Assume that for all $a, b \in F$, $f(a)g(b) - g(a)f(b) = 0$. Then for all $x \in F$, $f(x) = (g(x_0)^{-1}f(x_0))g(x)$. Since $g(x_0)^{-1}f(x_0)$ is a constant, S is linearly dependent.

(b) By part a), we need to find $a, b \in F$, such that $f(a)g(b) - g(a)f(b) \neq 0$. Since f and g are not identically zero, there exist $a, b \in F$, such that $f(a) \neq 0$ and $g(b) \neq 0$. Then, if $g(a) = 0$, $f(a)g(b) - g(a)f(b) = f(a)g(b) \neq 0$. Otherwise, if $g(a) \neq 0$, then $f(a)g(-a) - g(a)f(-a) = -f(a)g(a) - g(a)f(a) = -2f(a)g(a) \neq 0$.

9. (a) If $\alpha(ax) + \beta(by) + \gamma(cz) = 0$, then $(\alpha a)x + (\beta b)y + (\gamma c)z = 0$. Since $\{x, y, z\}$ is linearly independent, $\alpha a = \beta b = \gamma c = 0$. Now, since $abc \neq 0$, their inverses exist, to $\alpha = \beta = \gamma = 0$, that is $\{ax, by, cz\}$ is linearly independent.

   (b) Span of any set U is a linearly dependent set. Indeed, if $x \in U$, then $cx \in \text{span}(U)$ for any scalar c. Assume $c \neq 0$. Then $(-c) \cdot x + 1 \cdot (cx) = 0$ is a non-trivial combination of vectors from $\text{span}(U)$, so $\text{span}(U)$ is linearly dependent. Therefore, $\text{span}(\{x + z, x - y\})$ is a linearly dependent set.
   Note, $S' = \{x + z, x - y\}$ is a linearly independent set.

   (c) This is a linearly dependent set. Indeed, if $a(x + z) + b(x - y) + c(y + z) = 0$, then $(a + b)x + (-b + c)y + (a + c)z = 0$. Since $\{x, y, z\}$ is linearly independent, we get

$$\begin{array}{rcrcrcl} a & + & b & & & = & 0 \\ & - & b & + & c & = & 0 \\ a & & & + & c & = & 0. \end{array}$$

   This system of equations has a solution $(a, b, c) = (-c, c, c)$, for any $c \in F$. Therefore, for any $c \neq 0$, this is a non-trivial solution.

10. By definition, a set is linearly independent if it is not linearly dependent. Therefore, we will show that S is not linearly dependent.
    Assume that S is linearly dependent. Then there exist finitely many distinct vectors $f_i \in S$, $i = 1, 2, ..., n$, $n \in \mathbb{N}$, and scalars $\alpha_i \in F$ which are not all zero, such that $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n = 0$. Let k be the highest degree of polynomial with non-zero scalar, that is $k = max\{deg(f_i) \mid \alpha_i \neq 0, i = 1, 2, ..., n\}$ (note, this is a finite non-empty set, so maximum exists). Then, for some unique i_0, $f_i(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0$, $a_j \in F$, $j = 1, 2, ..., k$ and $a_k \neq 0$. We have that $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$ is a polynomial of degree k with leading coefficient $\alpha_{i_0} a_{i_0}$, since all polynomials in S have distinct degree smaller than k. However, $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n = 0$, so leading coefficient has to be zero, i.e. $\alpha_{i_0} a_{i_0} = 0$, implying $\alpha_{i_0} = 0$ since $a_{i_0} \neq 0$. This is a contradiction to our choice of k.

11. (a) Assume that $S_1 \cup S_2$ is a linearly dependent set. That is, for some $x_i \in S_1$, $i = 1, 2, ..., n$, and $y_j \in S_2$, $j = 1, 2, ..., m$, $n, m \in \mathbb{N}$, there exist $\alpha_i, \beta_j \in F$ not all zero, such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + \beta_1 y_1 + \beta_2 y_2 + \cdots \beta_m y_m = 0$, $n, m \geq 1$. Then, $v = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = -(\beta_1 y_1 + \beta_2 y_2 + \cdots \beta_m y_m) \in \text{span}(S_1) \cap \text{span}(S_2)$, so $v = 0$. Since S_1 and S_2 are linearly independent sets, this means that $\alpha_i = 0$ for $i = 1, 2, ..., n$ and $\beta_j = 0$ for $j = 1, 2, ..., m$, which is a contradiction.

    (b) Let $V = \mathbb{R}^2$ and consider $S_1 = \{(1, 1)\}$ and $S_2 = \{(0, 1), (1, 0)\}$.