MAT 240 - Problem Set 3

Due Thursday, October 9th

Questions 3a), 4a), 5b), 9c), 10 and 11a) will be marked.

1. Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$. Set $y = (0, 1, 1+i, -i)$ and $z = (1, i, 1, -i)$. Show that $x \in \text{span}(\{y, z\})$ if and only if $x_3 = (2-i)x_1 + (1+i)x_2$ and $x_4 = -(1+i)x_1 - i\,x_2$.

2. (§1.5, # 12) Let W be a nonempty subset of a vector space V over a field F. Prove that W is a subspace of V if and only if $\text{span}(W) = W$.

3. In each case, for the given subsets S_1 and S_2 of V, determine whether or not $\text{span}(S_1)$ is equal to $\text{span}(S_2)$. Justify your answers.

   a) Let $V = P_4(\mathbb{R})$, $S_1 = \{\, f \in V \mid f(1) = 0 \,\}$, $S_2 = \{\, x-1,\ x(x-1),\ x^2(x-1),\ x^3(x-1) \,\}$.

   b) $V = F^n$, where F is a field and $n \geq 3$, $S_1 = \{\, x,\ y \,\}$, $S_2 = \{\, x+z,\ y,\ z \,\}$ where x, y, $z \in V$ and $z \notin \text{span}(S_1)$.

   c) $V = M_{2\times 2}(\mathbb{F}_5)$,

$$S_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},\ \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix},\ \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \right\}$$
$$S_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + d = 0 (\text{mod}\, 5),\ b + c = 0 (\text{mod}\, 5) \right\}$$

4. Let $V = M_{2\times 2}(\mathbb{Q})$ be the vector space of 2×2 matrices with entries in the real numbers $\mathbb{Q}$.

   a) Find a subset S of V such that $\text{span}(S) = V$, and each matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in S has the property $a_{11} + a_{12} = 1$. (*Note*: Be sure to demonstrate that $\text{span}(S) = V$ for your choice of S.)

   b) Suppose that S' is a subset of V which has the property that each $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in S' satisfies $a_{11} + a_{12} = 0$. Prove that $\text{span}(S') \neq V$.

5. Let $V = P(\mathbb{F}_5)$ (the vector space of polynomials in one variable, with coefficients in the field $\mathbb{F}_5$).

   a) Find a set S of (nonzero) polynomials in V such that every polynomial in S has degree 2 and the coefficient of x is equal to 1, and $\text{span}(S) = P_2(\mathbb{F}_5)$. (*Note*: Here, $P_2(\mathbb{F}_5)$ is the subspace of V made up of polynomials of degree at most 2. Make sure that you prove that $\text{span}(S) = P_2(\mathbb{F}_5)$ for your choice of S.)

   b) Suppose that S' is a subset of V such that every $f(x) \in S'$ satisfies $f(1) = f(3)$. Prove that $\text{span}(S') \neq V$.

6. Let S_1 and S_2 be two subsets of a vector space V.

   a) Prove that $\text{span}(S_1 \cap S_2) \subset \text{span}(S_1) \cap \text{span}(S_2)$.

   b) Find an example of a vector space V and two nonempty subsets S_1 and S_2 of V that have the property that $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are not equal.

c) Find an example of a vector space V and two distinct nonempty subsets S_1 and S_2 of V that have the property that $\operatorname{span}(S_1 \cap S_2) = \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$.

7. Determine whether the subset S of the vector space V is linearly independent. (Justify your answer fully).

a) $V = M_{2\times 2}(\mathbb{C})$ and

$$S = \left\{ \begin{pmatrix} -1+i & 1+i \\ 0 & 1-i \end{pmatrix}, \begin{pmatrix} i+1 & 1 \\ -1 & i-1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ i & 2 \end{pmatrix} \right\}$$

b) $V = M_{2\times 2}(\mathbb{F}_3)$ and

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

c) $V = P(\mathbb{Q})$, $S = \{ x^5 + x^4 - x^3, x^5 + x^4 - x^2, x^5 + x^4 - x \}$.

8. Let F be a field and let $V = \{ f : F \to F \}$ be the vector space of all functions from F to F.

a) Prove that if $f(x)$ and $g(x) \in V$, then the set $S = \{ f(x), g(x) \}$ is linearly independent if and only if there exist a and $b \in F$ such that $f(a)g(b) - g(a)f(b) \neq 0$.

b) Assume that F has the property that $1 + 1 \neq 0$. Let $f(x) \in V$ be a nonzero function such that $f(-c) = f(c)$ for all $c \in F$, and let $g(x) \in V$ be a nonzero function such that $g(-c) = -g(c)$ for all $c \in F$. Prove that $\{ f(x), g(x) \}$ is linearly independent.

9. Suppose that x, y and z are distinct vectors in a vector space V over a field F, and $S = \{x, y, z\}$ is linearly independent. For each set S given below, determine whether S is linearly independent. Please justify your answers.

a) Let a, b and c be nonzero sclars (nonzero elements of F) and let $S = \{ ax, by, cz \}$.

b) Let $S = \operatorname{span}(\{ x + z, x - y \})$.

c) Let $S = \{ x + z, x - y, y + z \}$.

10. Let F be a field and let $V = P(F)$. Let S be a nonempty set of nonzero polynomials in V such that no two polynomials in S have the same degree. Prove that S is linearly independent. (*Note*: Do not assume that S is finite. Recall that if $f \in V$ is nonzero, then there exist an integer $n \geq 0$ and elements $a_0, \ldots, a_n \in F$ with $a_n \neq 0$ and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. The integer n is the degree of f.)

11. Let S_1 and S_2 be linearly independent subsets of a vector space V.

a) Suppose that $\operatorname{span}(S_1) \cap \operatorname{span}(S_2) = \{0\}$. Prove that $S_1 \cup S_2$ is linearly independent.

b) Give an example of two linearly independent subsets S_1 and S_2 (in some vector space) having the property that for every vector $x \in S_1$, the vector cx does not belong to S_2 for any scalar $c \in F$, and $S_1 \cup S_2$ is linearly dependent.