MAT 240, Fall 2008
Solutions to Problem Set 2

1. In this problem we need to determine if given V is a field.

(a) If $V = \{(a_1, a_2)|a_1, a_2 \in \mathbb{R}\}$, $F = \mathbb{R}$, $(a_1, a_2) + (b_1 + b_2) = (a_1 + b_1, a_2 + b_2)$ and $c(a_1, a_2) = (ca_1 + c - 1, ca_2 + c - 1)$, then V is not a field since axioms $(VS7)$ and $(VS8)$ fail. Indeed, $c((a_1, a_2) + (b_1, b_2)) = c(a_1 + b_1, a_2 + b_2) = (c(a_1 + b_1) + c - 1, c(a_2 + b_2) + c - 1)$, but $c(a_1, a_2) + c(b_1, b_2)) = (c(a_1 + b_1) + 2c - 1, 2(a_2 + b_2) + 2c - 2)$. If we take, for example, $c = 1$, $a_1 = a_2 = b_1 = b_2 = 1$, we get that $c((a_1, a_2) + (b_1, b_2)) = (5, 5) \neq (6, 6) = c(a_1, a_2) + c(b_1, b_2))$, so the property $(VS7)$ does not hold. Moreover, $(c+d)(a_1, a_2) = (ca_1 + da_1 + c + d - 1, ca_2 da_2 + c + d - 1)$ and $c(a_1, a_2) + d(a_1, a_2) = (ca_1 + da_1 + c + d - 2, ca_2 + da_2 + c + d - 2)$, which are not equal if we take, for example, $c = d = 1$, $a_1 = a_2 = 1$.

(b) If $V = P(\mathbb{C})$, $F = \mathbb{C}$, and $(f + g)(z) = f(z) + g(z)$, $(cf)(z) = f(cz)$ for $c \in \mathbb{C}$, then V is not a vector space over F since the axiom $(VS8)$ fails. Indeed, for any $c, d \in \mathbb{C}$, $((c+d)f)(z) = f((c+d)z)$ and $((cf)+(df))(z) = (cf)(z)+(df)(z) = f(cz)+f(dz)$. Take, for example, $f(z) = z^2$ and $c = d = 1$, then $f((c + d)z) = 4z^2 \neq 2z^2 = f(cz) + f(dz)$.

(c) If $V = \{f : \mathbb{Z} \to \mathbb{R}|f(n) > 0$ for all $n \in \mathbb{Z}\}$ with addition $(f + g)(n) = f(n)g(n)$ and scalar multiplication $(cf)(n) = (f(n))^c$, $c \in \mathbb{R}$, then V is a vector space over $\mathbb{R}$. We check all axioms in their order for any $f, g, h \in V$ and $c, d \in \mathbb{R}$:
(VS1) $(f + g)(n) = f(n)g(n) = g(n)f(n) = (g + f)(n)$, since multiplication is commutative in $\mathbb{R}$;
(VS2) $((f + g) + h)(n) = (f + g)(n)h(n) = (f(n)g(n))h(n) = f(n)(g(n)h(n)) = (f + (g + h))(n)$, since multiplication is associative in $\mathbb{R}$;
(VS3) zero vector is $\phi(n) = 1$ since $(f + \phi)(n) = f(n)\phi(n) = f(n) \cdot 1 = f(n)$ (note, $1 > 0$, so $\phi \in V$);
(VS4) let $(-f)(n) = \frac{1}{f(n)}$ which exists since $f(n) > 0$ and $\frac{1}{f(n)} > 0$ for all all $n \in \mathbb{Z}$, so $-f \in V$; then $(f + (-f))(n) = 1 = \phi(n)$;
(VS5) $(1 \cdot f)(n) = f^1(n) = f(n)$;
(VS6) $((cd)f)(n) = f^{cd}(n) = f^{dc}(n) = (f^d(n))^c = (cf^d)(n) = (c(d \cdot f))(n)$, since multiplication is commutative in $\mathbb{R}$;
(VS7) $(c(f + g))(n) = (f + g)^c(n) = (f(n)g(n))^c = f^c(n)g^c(n) = (cf + cg)(n)$;
(VS8) $((c + d)f)(n) = f^{c+d}(n) = f^c(n)f^d(n) = (cf + df)(n)$.

(d) Let $V = \{(a_1, a_2)|a_1, a_2 \in \mathbb{F}_5\}$, $F = \mathbb{F}_5$, $(a_1, a_2) + (b_1, b_2) = (a_1 b_1 (mod 5), a_2 + b_2 (mod 5))$ and $c(a_1, a_2) = (ca_1(mod 5), ca_2(mod 5))$, $c \in \mathbb{F}_5$. Then, V is not a vector space over F since axioms (VS4), (VS7) and (VS8) are not satisfied. We will assume that all operations are done modulo 5. The zero vector is $(1, 0)$ since for any $(a_1, a_2) \in V$, $(a_1, a_2) + (1, 0) = (a_1 \cdot 1, a_2 + 0) = (a_1, a_2)$. Then any vector of form $(0, a_2) \in V$ does not have the inverse since for any $(b_1, b_2) \in V$, $(0, a_2) + (b_1, b_2) = (0, a_2 + b_2) \neq (1, 0)$. Moreover, for any $c \in \mathbb{F}_5$, and any $(a_1, a_2), (b_1, b_2) \in V$, $c((a_1, a_2) + (b_1, b_2)) = (ca_1 b_1, ca_2 + cb_2)$ and $c(a_1, a_2) + c(b_1, b_2) = (ca_1, ca_2) + (cb_1, cb_2) = (c^2 a_1 b_1, ca_2 + cb_2)$. Hence, if we take, for example, $c = 2$, $a_1 = b_1 = 1$, $a_2 = b_2 = 0$, we get that $c((a_1, a_2) + (b_1, b_2)) = (2, 0) \neq$

$(4,0) = c(a_1, a_2) + c(b_1, b_2)$. Finally, if we take $c = d = 1$ and $(a_1, a_2) = (1, 0)$, we have that $(c+d)(a_1, a_2) = 2(1,0) = (2,0)$, but $c(a_1, a_2) + d(a_1, a_2) = 1(1,0) + 1(1,0) = (1,0)$. However, $(2,0) \neq (1,0)$, so axiom (VS8) fails.

(e) Let $V = \{f | f : \mathbb{R} \to \mathbb{R}\}$ and $F = \mathbb{R}$, $(f+g)(t) = \frac{1}{2}(f(t) + f(-t) + g(t) + g(-t))$ and $(cf)(t) = c \cdot f(t)$, $c \in \mathbb{R}$. Then V is not a vector space over F since axioms (VS3), (VS4), and (VS8) are not satisfied. If there were the zero vector ϕ then for $f(t) = t+1$, $(f+\phi)(t) = f(t)$. So, $\frac{1}{2}(f(t) + f(-t) + \phi(t) + \phi(-t)) = \frac{1}{2}(2 + \phi(t) + \phi(-t)) = t+1$, so $\frac{1}{2}(\phi(t) + \phi(-t)) = t$. But then also, if $g(t) = t^2 + 1$, we need that $(g+\phi)(t) = g(t)$. However, $(g+\phi)(t) = t^2 + 1 + \frac{1}{2}(\phi(t) + \phi(-t)) = t^2 + t + 1 \neq t^2 + 1$. Hence, there is no zero vector. If there is not zero vector, we cannot define the additive inverse. Also, if we take $c = d = 1$ and $f(t) = t$, we get that $((c+d)f)(t) = 2f(t) = 2t$ which is not equal to $(cf + df)(t) = \frac{1}{2}(ct - ct + dt - dt) \equiv 0$.

(f) Given $V = \{(z_1, z_2) | z_1, z_2 \in \mathbb{C}\}$, $F = \mathbb{C}$ with addition $(z_1, z_2) + (z_1', z_2') = (z_1 + z_1', z_2 + z_2')$ and scalar multiplication $c(z_1, z_2) = (cz_1, \bar{c}z_2)$ is a vector space. Since addition is inherited from $\mathbb{C}^2$, axioms $(VS1)$-$(VS4)$ hold, and we will not check them here.
(VS5) Multiplicative identity is 1 since for all $(z_1, z_2) \in V$, $1 \cdot (z_1, z_2) = (1 \cdot z_1, \bar{1} \cdot z_2) = (1 \cdot z_1, 1 \cdot z_2) = (z_1, z_2)$.
(VS6) For any $c, d \in \mathbb{C}$, $(cd)(z_1, z_2) = ((cd)z_1, \overline{cd}z_2) = (c(dz_1), \bar{c}(\bar{d}z_2)) = c(d(z_1, z_2))$.
(VS7) For any $c \in \mathbb{C}$, $c((z_1, z_2) + (z_1', z_2')) = c(z_1 + z_1', z_2 + z_2') = (c(z_1 + z_1'), \bar{c}(z_2 + z_2')) = (cz_1 + cz_1', \bar{c}z_2 + \bar{c}z_2') = (cz_1, \bar{c}z_2) + (cz_1', \bar{c}z_2') = c(z_1, z_2) + c(z_1', z_2')$.
(VS8) For any $c, d \in \mathbb{C}$, $(cd)(z_1, z_2) = ((cd)z_1, \overline{cd}z_2) = (c(dz_1), \bar{c}(\bar{d}z_2)) = c(dz_1, \bar{d}z_2) = c(d(z_1, z_2))$.

2. Let V be a vector space over a field F. We need to show that inverse of a vector is unique. Let $x \in V$ and assume that there exist $y_1, y_2 \in V$ such that $x + y_1 = x + y_2 = 0$. Then we have that

$$\begin{aligned}
y_1 &= y_1 + 0 && \text{(0 is the zero vector in } V\text{)} \\
&= y_1 + (x + y_2) && \text{(by our assumption)} \\
&= (y_1 + x) + y_2 && \text{(associativity of addition in } V\text{)} \\
&= (x + y_1) + y_2 && \text{(commutativity of addition in } V\text{)} \\
&= 0 + y_2 && \text{(by our assumption)} \\
&= y_2 + 0 && \text{(commutativity of addition in } V\text{)} \\
&= y_2 && \text{(0 is the zero vector in } V\text{).}
\end{aligned}$$

3. For a given subset $W \subset V$, where $V = \mathbb{R}^n$, $n \geq 3$, we need to determine if W is a subspace of V. In each part of the problem, we will demonstrate that requirements of Theorem 1.3 are either satisfied or not satisfied. Let $x = (a_1, a_2, \ldots, a_n)$, and $y = (b_1, b_2, \ldots, b_n)$.

(a) The subset $W = \{x \in V | a_1 + a_2 + \cdots + a_{n-1} = a_n\}$ is a subspace. Obviously, $0 = (0, 0, ..., 0)$ is in W since $0 + 0 + \cdots + 0 = 0$. Let $x, y \in W$. Then, $x + y = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ and $(a_1 + b_1) + (a_2 + b_2) + \cdots + (a_{n-1} + b_{n-1}) = (a_1 + a_2 + \cdots + a_{n-1}) + (b_1 + b_2 + \cdots + b_{n-1}) = a_n + b_n$, since $x, y \in W$. So $x + y \in W$. Finally, for above x and $c \in \mathbb{R}$, we have that $cx = (ca_1, ca_2, \ldots, ca_n)$ and $ca_1 + ca_2 + \cdots + ca_{n-1} = c(a_1 + a_2 + \cdots + a_{n-1}) = ca_n$, since $x \in W$. Hence $cx \in W$. Therefore, by Theorem 1.3, W is a subspace of V.

(b) If $W = \{x \in V | a_1^2 a_3 = -a_2\}$, then W is not a subspace of V since. Let $x \in V$. Then, $cx = (ca_1, ca_2, ca_3, \ldots, ca_n)$, and we know that $a_1^2 a_3 = -a_2$ since $x \in W$. However, $(ca_1)^2(ca_3) = c^3 a_1 a_3 \neq ca_2$ for, for example, $c = 2$, $a_1 = a_2 = a_3 = 1$.

(c) Let $W = \{x \in V | a_1 - \sqrt{3}a_2 = 4a_3\}$. Then W is a subspace of V. Indeed, $0 = (0, 0, ..., 0) \in W$ since $0 - \sqrt{3}0 = 0 = 4 \cdot 0$. Let $x, y \in W$. Then $x + y = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

and $(a_1+b_1)-\sqrt{3}(a_2+b_2)=(a_1-\sqrt{3}a_2)+(b_1-\sqrt{3}b_2)=4a_3+4b_3=4(b_1+b_3)$, so $x+y\in W$. Finally, for any $c\in\mathbb{R}$, $cx=(ca_1,ca_2,ca_3,\ldots,ca_n)$ and $(ca_1)-\sqrt{3}(ca_2)=c(a_1-\sqrt{3}a_2)=c(4a_3)=4(ca_3)$, so $cx\in W$. By Theorem 1.3, W is a subspace of V.

(d) If $W=\{x\in V|\sqrt{2}a_3\in\mathbb{Q}\}$, W is not a subspace of V. Indeed, W needs to be a vector space over $\mathbb{R}$. However, if $c=\sqrt{3}$, then $cx\notin W$ since $\sqrt{2}(ca_3)=\sqrt{3}(\sqrt{2}a_3)\notin\mathbb{Q}$ because $\sqrt{2}a_3\in\mathbb{Q}$.

(e) If $W=\{x\in V|a_1\leq 1\}$, W is not a subspace of V. Let $x\in W$ be such that $0<a_1<1$. Then take $c>\frac{1}{a_1}\in\mathbb{R}$. Then $ca_1>\frac{1}{a_1}a_1>1$, so $cx\notin W$.

4. Let $V=P(\mathbb{C})$. For given $W\subset V$, we need to determine if W is a subspace of V.

(a) If $W=\{f\in V|f(1+i)=if(1-i)\}$, W is a subspace of V. Obviously, zero vector of V is $\zeta(z)\equiv 0$ and $\zeta(1+i)=0=i\zeta(1-i)$, so $\zeta\in W$. Assume that $f,g\in W$. Then $(f+g)(1+i)=f(1+i)+g(1+i)=if(1-i)+ig(1-i)=i((f+g)(1-i))$. Hence, $f+g\in W$. Given $c\in\mathbb{C}$, $(cf)(1+i)=c\cdot f(1+i)=c\cdot if(1-i)=i((cf)(1-i))$, implying that $cf\in W$. Therefore, by Theorem 1.3, W is a subspace of V.

(b) If $W=\{f\in V|f(i)^2=f(-1)^2\}$, then W is not a subspace of V. To show this, let $f(z)=1$ and $g(z)=z^2$. Then $f(i)^2=1=f(-1)^2$ and $g(i)^2=(-1)^2=1=g(-1)^2$, so $f,g\in W$. However, $f+g\notin W$. Indeed, $(f+g)(z)=1+z^2$, so $(f+g)(i)=1+i^2=0$ and $(f+g)(-1)=1+(-1)^2=2$. But $0^2\neq 2^2$.

(c) Let $W=\{f\in V|f(iz)=if(z)+f(-z)\}$. To prove that W is a subspace of V, first we check that zero function, which is the zero vector in $P(\mathbb{C})$ belongs to W. Indeed, if $\phi(z)=0$ for all $z\in\mathbb{C}$, then $\phi(iz)=0=i\cdot 0+0=i\phi(z)+\phi(-z)$. Hence, $\phi\in W$. Then, assume that $f,g\in W$, that is $f(iz)=if(z)+f(-z)$ and $g(iz)=ig(z)+g(-z)$ for all $z\in\mathbb{C}$. Then, $(f+g)(iz)=f(iz)+g(iz)=(if(z)+f(-z))+(ig(z)+g(-z))=i(f(z)+g(z))+(f(-z)+g(-z))=i((f+g)(z))+(f+g)(-z)$, so $f+g\in W$. Finally, let $c\in\mathbb{C}$. Then $(cf)(iz)=c\cdot f(iz)=c(if(z)+f(-z))=icf(z)+cf(-z)=i((cf)(z))+(cf)(-z)$, which implies that $cf\in W$. By Theorem 1.3, W is a subspace of V.

(d) If $W=\{f\in V|f(1)-\overline{f(i)}=0\}$, then W is not a subspace of V. To see this, let $f(z)=(z-1)(z-i)+1$. It is easy to check that $f\in W$. Let $c=i$. Then $(cf)(1)-\overline{(cf)(i)}=i-\bar{i}=i-(-i)=2i\neq 0$. Hence, $cf\notin W$.

(e) If $W=\{f\in V|f(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0,$ and $a_j=0$ for all j even: $a_0=a_2=a_4=a_6=\cdots=0\}$, then W is a subspace of V. Obviously, zero function is in W. Without loss of generality, let $n\geq m$ and $f(z)=a_{2m+1}z^{2m+1}+a_{2m-1}z^{2m-1}+\cdots+a_3z^3+a_1z$ and $g(z)=b_{2n+1}z^{2n+1}+b_{2n-1}z^{2n-1}+\cdots+b_{2m+1}z^{2m+1}+b_{2m-1}z^{2m-1}+\cdots+b_3z^3+b_1z$, where $a_{2i+1}\in\mathbb{C}$, for $i=0,1,\ldots,m$, $b_{2j+1}\in\mathbb{C}$, for $j=0,1,\ldots,n$. Then $f,g\in W$ and $(f+g)(z)=b_{2n+1}z^{2n+1}+b_{2n-1}z^{2n-1}+\cdots+(a_{2m+1}+b_{2m+1})z^{2m+1}+(a_{2m-1}+b_{2m-1})z^{2m-1}+\cdots+(a_3+b_3)z^3+(a_1+b_1)z$, so $f+g\in W$. Also, $(cf)(z)=(ca_{2m+1})z^{2m+1}+(ca_{2m-1})z^{2m-1}+\cdots+(ca_3)z^3+(ca_1)z$, so $cf\in W$ when $c\in\mathbb{C}$.

5. Given W_1 and W_2 subspaces of a vector space V, we need to prove that $W_1\cup W_2=\{x\in V|x\in W_1$ of $x\in W_2\}$ if and only if $W_1\subset W_2$ or $W_2\subset W_1$.
$\Leftarrow$: Without loss of generality, assume that $W_2\subset W_1$. Then, $W_1\cup W_2=W_1$ which is a subspace of V.
$\Rightarrow$: We will prove this direction by contra-position. That is, if W_1 is not a subset of W_2 and W_2 is not a subset of W_1, then $W_1\cup W_2$ does not have to be a subspace of V. Indeed, if $W_1\not\subset W_2$, there exists $w_1\in W_1$ such that $w_1\notin W_2$, and if $W_2\not\subset W_1$, there exists $w_2\in W_2$

such that $w_2 \notin W_1$. Assume that $W_1 \cup W_2$ is a subspace of V. Then $w_1, w_2 \in W_1 \cup W_2$, so $w_1 + w_2 \in W_1 \cup W_2$, so $w_1 + w_2 \in W_1$ or $w_1 + w_2 \in W_2$. Without loss of generality, assume that $w_1 + w_2 = a \in W_1$. Then $w_2 = a + (-w_1)$. Since W_1 is a subspace of V, inverse of w_1, $-w_1$, belongs to W_1, as well as sum of two vectors from W_1. So, $w_2 = a + (-w_1) \in W_1$, which is a contradiction to $w_2 \in W_2 \backslash W_1$. Therefore, $W_1 \cup W_2$ is not a subspace of V.

6. Let W_1 and W_2 be subspaces of a vector space V.

   (a) Then $W_1 + W_2 = \{x + y | x \in W_1, y \in W_2\}$ is a subspace of V. Indeed, $0 \in W_1$ and $0 \in W_2$ since W_1 and W_2 are subspaces of V. Hence, $0 + 0 = 0 \in W_1 + W_2$. Let $x_1, x_2 \in W_1$ and $y_1, y_2 \in W_2$. Then $x_1 + y_1, x_2 + y_2 \in W_1 + W_2$ and $(x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) \in W_1 + W_2$ since $x_1 + x_2 \in W_1$ and $y_1 + y_2 \in W_2$, as W_1 and W_2 are subspaces of V. Also, if c is a scalar, $c(x_1 + y_1) = (cx_1 + cy_1) \in W_1 + W_2$ since $cx_1 \in W_1$ and $cy_1 \in W_2$. Therefore, by Theorem 1.3, $W_1 + W_2$ is a subspace of V. Since $0 \in W_1$, $W_2 = \{y = 0 + y | y \in W_2\} \subset W_1 + W_2$. Similarly, $W_1 \subset W_1 + W_2$.

   (b) Let Z be a subspace of V such that $W_1 \subset Z$ and $W_2 \subset Z$. We want to show that $W_1 + W_2 \subset Z$. Indeed, since $W_1 \subset Z$, for any $x \in W_1$, also $x \in Z$. Similarly, for any $y \in W_2$, also $y \in Z$. Hence, $x + y \in Z$ because Z is a subspace of V. But this means that $W_1 + W_2 = \{x + y | x \in W_1, y \in W_2\} \subset Z$.