MAT 240 - Problem Set 2

Due Thursday, October 2

Questions 1c), 1d), 1e), 3a), 4b), 4c), and 5 will be marked.

1. In each of the following parts, determine whether or not the set V with the given operations of addition and scalar multiplication, is a vector space over the given field F. If so, show that each of the axioms holds. If not, say which axioms fail to hold (and explain why).
   a) $V = \{ (a_1, a_2) \mid a_1, a_2 \in \mathbb{R} \}$, $F = \mathbb{R}$, with vector sum $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ and scalar multiplication $c(a_1, a_2) = (ca_1 + c - 1, ca_2 + c - 1)$.
   b) $V = P(\mathbb{C})$ (the space of polynomial functions over $\mathbb{C}$), $F = \mathbb{C}$, with vector sum $(f+g)(z) = f(z) + g(z)$, and scalar multiplication $(cf)(z) = f(cz)$, $c \in \mathbb{C}$.
   c) Let V be the set of functions $f : \mathbb{Z} \to \mathbb{R}$ such that $f(n) > 0$ for every integer n, with vector sum $(f + g)(n) = f(n)g(n)$ and scalar multiplication $(cf)(n) = (f(n))^c$, $c \in \mathbb{R}$. (Notice that if t is a positive real number, then $t = e^u$ for some real number u, and if c is a real number, then $t^c = e^{cu}$.)
   d) $V = \{ (a_1, a_2) \mid a_1, a_2 \in \mathbb{F}_5 \}$, $F = \mathbb{F}_5$, with vector sum $(a_1, a_2) + (b_1, b_2) = (a_1 b_1 (\mathrm{mod}\, 5), (a_2 + b_2)(\mathrm{mod}\, 5))$, and scalar multiplication $c(a_1, a_2) = (ca_1 (\mathrm{mod}\, 5), ca_2 (\mathrm{mod}\, 5))$, $c \in \mathbb{F}_5$.
   e) Let V be the set of functions $f : \mathbb{R} \to \mathbb{R}$, $F = \mathbb{R}$, with vector sum $(f + g)(t) = \frac{1}{2}(f(t) + f(-t) + g(t) + g(-t))$, and scalar multiplication $(cf)(t) = c \cdot f(t)$, $c \in \mathbb{R}$.
   f) Let $V = \{ (z_1, z_2) \mid z_1, z_2 \in \mathbb{C} \}$, $F = \mathbb{C}$, with vector sum $(z_1, z_2) + (z_1', z_2') = (z_1 + z_1', z_2 + z_2')$ and scalar multiplication $c(z_1, z_2) = (cz_1, \bar{c} z_2)$, $c \in \mathbb{C}$. (Here, $\bar{c}$ is the complex conjugate of c.)

2. Let V be a vector space over a field F. Using properties of fields and the axioms in the definition of a vector space, prove that for each choice of vector $x \in V$, the element $y \in V$ such that $x + y = \mathbf{0}$ is unique.

3. Assume that $n \in \mathbb{Z}$ and $n \geq 3$. Which of the following sets W of vectors $x = (a_1, a_2, \ldots, a_n) \in V = \mathbb{R}^n$. are subspaces of V? If W is a subspace of V, prove it. If not, demonstrate how some property of subspace is not satisfied.
   a) $W = \{ x \in V \mid a_1 + a_2 + \cdots + a_{n-1} = a_n \}$
   b) $W = \{ x \in V \mid a_1^2 a_3 = -a_2 \}$
   c) $W = \{ x \in V \mid a_1 - \sqrt{3} a_2 = 4a_3 \}$
   d) $W = \{ x \in V \mid \sqrt{2}\, a_3 \in \mathbb{Q} \}$. (Here, $\mathbb{Q}$ is the field of rational numbers.)
   e) $W = \{ x \in V \mid a_1 \leq 1 \}$

4. Let $V = P(\mathbb{C})$ be the complex vector space of polynomials in one variable with complex coefficients. Which of the following subsets W of V is a subspace of V? If the subset is a subspace, prove it. If not, demonstrate how some property of subspace is not satisfied by the subset.
   a) $W = \{ f \in V \mid f(1 + i) = i\, f(1 - i) \}$
   b) $W = \{ f \in V \mid f(i)^2 = f(-1)^2 \}$
   c) $W = \{ f \in V \mid f(iz) = \overline{i\, f(z)} + f(-z) \text{ for all } z \in \mathbb{C} \}$
   d) $W = \{ f \in V \mid f(1) - \overline{f(i)} = 0 \}$
   e) $W = \{ f \in V \mid f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, and $a_j = 0$ for j even : $a_0 = a_2 = a_4 = a_6 = \cdots = 0 \}$

5. Let W_1 and W_2 be subspaces of a vector space V. Let $W_1 \cup W_2 = \{ x \in V \mid x \in W_1 \text{ or } x \in W_2 \}$. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subset W_2$ or $W_2 \subset W_1$.

6. §1.3, #23.