

MAT 240 - Problem Set 1

Due Thursday, September 25th

NOTE:

- Questions 4, 5, 6a), 9b) and 10c) will be marked. It is not necessary to hand in solutions to other questions, but it is recommended that students work through at least two thirds of the questions in order to learn the material thoroughly.
- In order to receive full marks for computational questions (for example, questions 1, 2, 3a), and 4), all details of the computation must be included in the solution. Even if the correct final answer is given, marks will be deducted if some details are left out.
- In solving questions involving proofs, *unless the question specifically lists which results may be used (see question 5 below)*, it is not necessary to reprove facts that have been proved in class, in notes posted on the course home page, in the text (from sections of the text that have been covered in class), or on a previous problem set.

1. Express the following complex numbers in the form $a + bi$, $a, b \in \mathbb{R}$.

- a) $\frac{(2-i)^3}{(4i+1)^2}$
b) $\frac{1}{\sqrt{5}+i} - \frac{2}{3-i}$

2. In each case below, find all complex numbers that belong to the indicated set S . Sketch the set S .

- a) $S = \{z \in \mathbb{C} \mid z^4 \in \mathbb{R}\}$.
b) $S = \{z \in \mathbb{C} \mid z^3 \in i\mathbb{R}\}$.

3. Let $F = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$. If $(a_1, a_2), (b_1, b_2) \in F$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) \quad (\text{addition in } F)$$
$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 + 3 a_2 b_2, a_2 b_1 + a_1 b_2) \quad (\text{multiplication in } F)$$

- a) Find a multiplicative inverse of the element $(\sqrt{5}, 1)$ in F .
b) Prove that F is not a field relative to the above addition and multiplication by finding an example of a nonzero element of F which does not have a multiplicative inverse. (*Hint:* Use the fact that 3 is a square in the real numbers.)
4. Let \mathbb{F}_5 be the finite field containing 5 elements. That is, $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$, with addition and multiplication modulo 5. Let $F = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{F}_5\}$. If $(a_1, a_2), (b_1, b_2) \in F$, define

$$(a_1, a_2) + (b_1, b_2) = ((a_1 + b_1)(\text{mod } 5), (a_2 + b_2)(\text{mod } 5)) \quad (\text{addition in } F)$$
$$(a_1, a_2) \cdot (b_1, b_2) = ((a_1 b_1 + 2 a_2 b_2)(\text{mod } 5), (a_2 b_1 + a_1 b_2)(\text{mod } 5)) \quad (\text{multiplication in } F)$$

- a) Find an additive identity (a zero element) and a multiplicative identity in F .
b) Find an additive inverse of $(2, 1)$ in F .
c) Find a multiplicative inverse of $(1, 2)$ in F .
d) Find all elements (a_1, a_2) in F that satisfy $(a_1, a_2)^2 = (1, 1)$.
5. Suppose that a is an element of a field F and a satisfies $a = a^{-1}$. Using only the field axioms, along with with the additional property $(-1)c = -c$ for $c \in F$, prove that $a = 1$ or $a = -1$.

Use one axiom at a time, and indicate which axiom is being used (for example, existence of additive inverse).

6. In each case below, find all zeros of the indicated polynomial $f(x)$ in the field F .
 - a) $F = \mathbb{F}_7$, $f(x) = x^2 + 6x + 1$
 - b) $F = \mathbb{F}_5$, $f(x) = x^3 + 1$
 - c) $F = \mathbb{C}$, $f(x) = x^4 - i$
7. Let $F = \{0, 1, a, b\}$ be a field containing 4 elements. Assume that $1 + 1 = 0$. Prove that $b = a^{-1} = a^2 = a + 1$. (*Hint*: For example, for the first equality, show that $a \cdot b$ cannot equal 0, a , or b .)
8. Let p be an odd prime number. Let \mathbb{F}_p be the finite field of order p .
 - a) Prove that if n is an integer that is not divisible by p , then $r_p(n) \neq r_p(-n)$. (Here, $r_p(n)$ is the unique integer between 1 and $p - 1$ that has the property that $n - r_p(n)$ is divisible by p .)
 - b) Use part a) to show that if a is a nonzero element of \mathbb{F}_p , then $a \neq -a$.
 - c) Suppose that a and b belong to a field F , are both nonzero, and $a \neq \pm b$. Prove that $a^2 \neq b^2$.
 - d) An element a of a field F is said to be a *square* in F if $a = b^2$ for some $b \in F$. Prove that exactly $(p + 1)/2$ elements of the field \mathbb{F}_p are squares in \mathbb{F}_p . (*Hint*: Note that $0^2 = 0$ is a square, so the question amounts to showing that $(p - 1)/2$ of the nonzero elements of \mathbb{F}_p are squares. Consider the function $f(x) = x^2$ from \mathbb{F}_p to \mathbb{F}_p , and use other parts of the question to count the number of nonzero elements in the range of the function f .)
9. For each of the following sets, determine whether or not the set is a field, relative to the indicated addition and multiplication. If the set is a field, verify that all of the axioms in the definition are satisfied. If the set is not a field, demonstrate how one of the axioms fails to hold.
 - a) $F = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$. Assume that the multiplication and addition are as in \mathbb{R} .
 - b) $F = \{a + b\sqrt{7} \mid a, b \in \mathbb{Z}\}$. Assume that the multiplication and addition are as in \mathbb{R} . (Here, \mathbb{Z} is the set of integers).
 - c) $F = \{(a, b) \mid a, b \in \mathbb{F}_5\}$, with $(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2 - b_1b_2, a_2b_1 + a_1b_2)$ and $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$.
 - d) $F = \{(a, b) \mid a, b \in \mathbb{F}_5\}$, with $(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2 + 3b_1b_2, a_2b_1 + a_1b_2)$ and $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$.
10. A element a of a field F is a *cube* in F if there exists $b \in F$ such that $b^3 = a$.
 - a) Find an odd prime p for which every element in \mathbb{F}_p is a cube in \mathbb{F}_p . (Explain your answer.)
 - b) Find an odd prime p having the property that the set of cubes in \mathbb{F}_p is $\{0, 1, p - 1\}$. (Explain your answer.)
 - c) Suppose that p is an odd prime having the property that if a and b belong to \mathbb{F}_p and $a \neq b$, then $a^2 + ab + b^2 \neq 0$. Prove that every element of \mathbb{F}_p is a cube in \mathbb{F}_p .