Mat 240 : Notes on bases and dimension

(Material related to Section 1.6 of the text.)

Throughout these notes, assume that V is a vector space over a field F. Some useful results that appear in the text and not in these notes: Theorem 1.8; Theorem 1.11; Corollary appearing after Example 19.

Lemma 1. *Let S be a subset of V and let S_0 be a subset of V such that $V = \operatorname{span}(S_0)$. If $V \neq \operatorname{span}(S)$, then there exists $x \in S_0$ such that $x \notin \operatorname{span}(S)$.*

Proof. Suppose that $S_0 \subset \operatorname{span}(S)$. Then, according to an earlier theorem (Theorem 1.5 in the text: it says that if $S' \subset W$ and W is a subspace of V, then $\operatorname{span}(S') \subset W$), since $\operatorname{span}(S)$ is a subspace, we must have $\operatorname{span}(S_0) \subset \operatorname{span}(S)$. But $V = \operatorname{span}(S_0)$, by asssumption. Therefore $V \subset \operatorname{span}(S)$. Since $\operatorname{span}(S) \subset V$ (by definition), we must have $V = \operatorname{span}(S)$. This contradicts one of the assumptions of the lemma. Therefore it is not possible to have $S_0 \subset \operatorname{span}(S)$. Thus there exists at least one $x \in S_0$ such that $x \notin \operatorname{span}(S)$. *qed*

The next lemma is one half of Theorem 1.7 in Section 1.5 of the text.

Lemma 2. *Let S be a linearly independent subset of V such that $V \neq \operatorname{span}(S)$. If $x \in V$ and $x \notin \operatorname{span}(S)$, then the union $S \cup \{x\}$ is linearly independent.*

Proof. Suppose that $x_1, x_2, \ldots, x_n \in S$ are distinct vectors in S, $c_1, c_2, \ldots, c_{n+1} \in F$, and

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n + c_{n+1}x = \mathbf{0}.$$

We need to show that $c_1 = c_2 = \cdots = c_{n+1} = 0$. First, suppose that $c_{n+1} = 0$. Then we have

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = \mathbf{0},$$

and, from linear independence of S, we must have $c_1 = c_2 = \cdots = c_n = 0$.

To finish the proof of the lemma, we must show that c_{n+1} cannot be nonzero. Suppose that $c_{n+1} \neq 0$. Then, solving for x in terms of the vectors $x_1, \ldots, x_n$, we get

$$x = -c_{n+1}^{-1}c_1x_1 - c_{n+1}^{-1}c_2x_2 - \cdots - c_{n+1}^{-1}c_nx_n.$$

This tells us that $x \in \operatorname{span}(S)$ (because $x_1, \ldots, x_n \in S$). But we assumed that $x \notin \operatorname{span}(S)$. Therefore, the assumption $c_{n+1} \neq 0$ was incorrect. The only possibility is $c_{n+1} = 0$. We showed above that $c_{n+1} = 0$ implies $c_1 = \cdots = c_n = 0$. Therefore the set $S \cup \{x\}$ is linearly independent. *qed*

Theorem 1. *Suppose $S \subset V$ is linearly independent, and $S_0 \subset V$ is a finite set such that $\operatorname{span}(S_0) = V$. Then there is a subset S_1 of S_0 such that $S \cup S_1$ is a basis of V.*

Proof. Suppose that $\operatorname{span}(S) = V$. By assumption, S is linearly independent, so we have that S is linearly independent and spans (generates) V. That is, S is a basis of V.

Suppose that $\operatorname{span}(S) \neq V$. Apply Lemma 1 to conclude that there exists $x_1 \in S_0$ such that $x_1 \notin \operatorname{span}(S)$. Then apply Lemma 2 to conclude that since S was assumed to be linearly independent, and $x_1 \notin \text{span}(S)$, the set $S \cup \{ x_1 \}$ is linearly independent. If $\text{span}(S \cup \{ x_1 \}) = V$, then $S \cup \{ x_1 \}$ is a linearly independent set that spans V, and the conclusion of the theorem holds if we take $S_1 = \{ x_1 \}$.

If $\text{span}(S \cup \{ x_1 \}) \neq V$, then we apply Lemma 1, relative to the set $S \cup \{ x_1 \}$ (instead of S) to conclude that there exists $x_2 \in S_0$ such that $x_2 \notin \text{span}(S)$. Then, according to Lemma 2, we have that $(S \cup \{ x_1 \}) \cup \{ x_2 \} = S \cup \{ x_1, x_2 \}$ is linearly independent. If $\text{span}(S \cup \{ x_1, x_2 \}) = V$, then the conclusion of the theorem holds if we take $S_1 = \{ x_1, x_2 \}$. Otherwise, we apply Lemmas 1 and 2 to produce $x_3 \in S_0$ such that $S \cup \{ x_1, x_2, x_3 \}$ is linearly independent. Because the set S_0 is finite, this process cannot continue beyond finitely many steps. That is, there exists a subset $S_1 = \{ x_1, x_2, \dots x_m \}$ of S_0 such that $S \cup S_1$ is linearly independent and spans V. Thus $S \cup S_1$ is a basis of V.*qed*

The next theorem is related to Theorem 1.8 of Section 1.6 of the text.

Theorem 2. *A finite-dimensional vector space V has a finite basis.*

Proof. Take $S = \emptyset$ and S_0 a finite subset of V such that $V = \text{span}(S_0)$. Apply Theorem 1 to conclude that there exists a subset S_1 of S_0 such that $S \cup S_1$ is a basis of V. Note that $S \cup S_1 = S_1 \subset S_0$, so S_1 is a finite basis of V. *qed*

Theorem 3. *Suppose that $V = \text{span}(S_0)$ for a subset S_0 of V that contains exactly n vectors. If S is a linearly independent subset of V, then S contains at most n vectors.*

Proof. Assume that $S_0 = \{ x_1, \dots, x_n \}$. Let $S = \{ y_1, \dots, y_m \}$ be a linearly independent set containing exactly m vectors. Because $\text{span}(S_0) = V$, we have

$$y_1 = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

for some scalars $a_1, \dots a_n \in F$. Since any set containing the zero vector is linearly dependent, and S is linearly independent and contains y_1, we know that $y_1 \neq \mathbf{0}$. This means at least one a_j is nonzero. After renumbering the vectors in S_0, we can assume that $a_1 \neq 0$. In this case, we have

$$x_1 = a_1^{-1} y_1 - a_1^{-1} a_2 x_2 - \cdots - a_1^{-1} a_n x_n \in \text{span}(\{ y_1, x_2, \dots, x_n \}).$$

Clearly $x_2, \dots, x_n \in \text{span}(\{ y_1, x_2, \dots, x_n \})$. So we have

$$V = \text{span}(\{ x_1, \dots, x_n \}) \subset \text{span}(\{ y_1, x_2, \dots, x_n \}).$$

It follows that $V = \text{span}(\{ y_1, x_2, \dots, x_n \})$.

Next, write $y_2 = b_1 y_1 + c_2 x_2 + \cdots + c_n x_n$ for some scalars $b_1, c_2, \dots, c_n$. If $c_2 = \cdots = c_n = 0$, then $y_2 = b_1 y_1$ implies that $\{ y_1, y_2 \}$ is linearly dependent. But any subset of a linearly independent set is linearly independent. Since $\{ y_1, y_2 \}$ is a subset of the linearly independent set S, we have that $\{ y_1, y_2 \}$ is linearly independent. Therefore at least one c_j must be nonzero. After renumbering the vectors $x_2, \dots, x_n$, we may assume that $c_2 \neq 0$. Then we have

$$x_2 = c_2^{-1} y_2 - c_2^{-1} b_1 y_1 - c_2^{-1} c_3 x_3 - \ldots - c_2^{-1} c_n x_n,$$

which tells us that $x_2 \in \operatorname{span}(\{ y_1, y_2, x_3, \ldots, x_n \})$. Clearly, $y_1, x_3, \ldots, x_n \in \operatorname{span}(\{ y_1, y_2, x_3, \ldots, x_n \})$. So we have

$$V = \operatorname{span}(\{ y_1, x_2, \ldots, x_n \}) \subset \operatorname{span}(\{ y_1, y_2, x_3, \ldots, x_n \}).$$

That is, $V = \operatorname{span}(\{ y_1, y_2, x_3, \ldots, x_n \})$.

If $m > n$, this procedure continues until all the vectors $x_1, \ldots, x_n$ are replaced by the vectors $y_1, \ldots, y_n$, and we have $V = \operatorname{span}(\{ y_1, \ldots, y_n \})$. Now $y_{n+1} \in V$, so we then have $y_{n+1} \in \operatorname{span}(\{ y_1, \ldots, y_n \})$. That is, $y_{n+1} = d_1 y_1 + \cdots + d_n y_n$ for some scalars $d_1, \ldots, d_n \in F$. This can be rewritten as:

$$d_1 y_1 + d_2 y_2 + \cdots + d_n y_n + (-1) y_{n+1} = \mathbf{0}.$$

Since the last coefficient is -1 (which is nonzero), and $y_1, \ldots, y_{n+1}$ are distinct vectors in S, this tells us that S is linearly dependent. But we assumed that S is linearly independent. Therefore it is not possible to have $m > n$. *qed*

Corollary 1. *Suppose that V is a finite-dimensional vector space. Then any two bases of V contain the same number of vectors.*

Proof. Let S_1 be a finite basis of V. Let n be the number of (distinct) vectors in S_1. Let S_2 be another basis of V. Because S_2 is linearly independent, $V = \operatorname{span}(S_1)$, and S_1 contains n vectors, Theorem 3 tells us that S_2 contains at most n vectors. Because S_1 is linearly independent, and $V = \operatorname{span}(S_2)$, Theorem 3 tells us that n (the number of vectors in S_1) is less than or equal to the number of vectors in S_2. Therefore S_2 cannot contain fewer than n vectors. The only possibility is therefore that S_2 contains exactly n vectors. *qed*

Definition. The *dimension* of a (finite-dimensional) vector space V is the number of vectors in a basis of V. This is written as $\dim(V)$.

Theorem 4. *Let n be a positive integer. Let V be a vector space of dimension n.*

(1) If S is a linearly independent subset of V and S contains n (distinct) vectors, then S is a basis of V.

(2) If S' is a subset of V such that $V = \operatorname{span}(S')$ and S' contains n vectors, then S' is a basis of V.

Proof. For (1), note that since S is assumed to be linearly independent, in order to prove that S is a basis of V, we only need to prove that $V = \operatorname{span}(S)$. If $V \neq \operatorname{span}(S)$, then, taking any basis S_0 of V and applying Theorem 1, there exists a nonempty subset S_1 of S_0 such that $S \cup S_1$ is a basis of V. Now $S \cup S_1$ contains at least $n+1$ vectors (because S contains n vectors, S_1 contains at least one vector, and $S \cup S_1$ is linearly independent). According to Corollary 1, it is not possible for a set containing strictly more than n vectors to be a basis of V. Therefore the assumption that $V \neq \operatorname{span}(S)$ was false.

For (2), note that since we have assumed $V = \operatorname{span}(S')$, in order to prove that S' is a basis of V, we only need to prove that S' is linearly independent. Apply Theorem 1 with $S = \emptyset$ and $S_0 = S'$ to conclude that there exists a subset $\dot{S}$ of S' that is a basis of V. If S' is linearly dependent, then we cannot have $\dot{S} = S'$ (because $\dot{S}$ is linearly independent), so $\dot{S}$ is a basis of V containing at most $n-1$ vectors. This is impossible, because according to Corollary 1, every basis of V contains n vectors. Therefore it is impossible for S' to be linearly dependent. *qed*